SEP 13 1926

ELEVENTH ANNUAL MEETING OF THE OPTICAL SOCIETY OF AMERICA
PHILADELPHIA, PA., OCTOBER 21, 22 and 23, 1926.

Volume 13 Number 2

Journal of the Optical Society of America and Review of Scientific Instruments

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Published Monthly by the
OPTICAL SOCIETY OF AMERICA

OFFICE OF PUBLICATION
GEORGE BANTA PUBLISHING COMPANY
450-454 Ahnaip St.,
MENASHA, WISCONSIN

Entered as second-class matter May 31, 1921, at the postoffice at Menasha, Wis., under the Act of March 3, 1879. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 412, P. L. and R. authorized June 19, 1925.

Journal
of the
Optical Society of America
and
Review of Scientific Instruments

Published monthly by the Optical Society of America in cooperation with the Association of Scientific Apparatus Makers of the United States of America. The Section on Optics is devoted to original papers on experimental, theoretical, and applied optics. The Section on Scientific Instruments is devoted to original papers on instruments of all kinds (electrical, mechanical, thermal, etc. as well as optical) for research, instruction, and industrial processes. Manuscripts for publication are invited, whether authors are members of the Optical Society or not.

Annual subscription $5.00, dating from January of current year. Single copies $0.60. Back numbers: Vols. 1-5 (The Journal of the Optical Society, 1917–1921 inclusive) $16.00. Vol. 6–7, $5.00 per volume; Vol. 8—, $2.50 per volume. "Address publishers, George Banta Publishing Company, 450–454 Ahnaip Street, Menasha, Wisconsin, or F. K. Richtmyer, Manager, Cornell University, Ithaca, New York, for all correspondence regarding subscriptions, back numbers, changes of address, advertising rates and accounts, and missing numbers."

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Journal
of the
Optical Society of America
and
Review of Scientific Instruments

Vol. 13 AUGUST, 1926 Number 2

THE COMPUTATION OF COLORIMETRIC PURITY II

APPLICATION OF THE PURITY FORMULA TO NON-SPECTRAL COLORS*

By Irwin G. Priest

In a previous paper[1] I have given the derivation of the following formula for the colorimetric puri y of spectral colors[2]:

$$p=\frac{1-\dfrac{c}{L_r r+L_g g+L_b b}}{1-\dfrac{C}{L_r R+L_g G+L_b B}}.$$

In that paper, nothing was said concerning the purity of non-spectral colors ("purples"). The purpose of the present paper is to elucidate

* Partial Report No. 5 on cooperative investigations in colorimetry by the Bureau of tandards and the Munsell Research Laboratory. Published by permission of the Director of the Bureau of Standards, U.S. Department of Commerce, and the Director of the Munsell Research Laboratory.

A much more comprehensive discussion of the purity formulas by Deane B. Juddis published simultaneously with the present paper. As one item of this more general treatment, Mr. Judd independently arrives at a result in accord with the special formula given in the present paper. The methods of approach to the problem are, however, quite different in the two cases. Both demonstrations are published because it will probably be helpful to the reader to follow these independent arguments leading to the same conclusion. Depending upon their own previously fixed modes of thought, some readers will probably find it easier to follow one argument and some, the other.

[1] J.O.S.A. & R.S.I., *9*, pp. 503–520; Nov. 1924. It will be necessary for the reader to refer to this paper in order to understand what follows.

[2] That is, colors having actual dominant wave lengths occurring in the visible spectrum; or more specifically, colors which can be evoked by a suitable mixture of a homogeneous stimulus and the normal neutral heterogeneous stimulus ("white light"). "Spectrum colors" will be understood as spectral colors of unit purity.

the application of the above mentioned formula[3] to the computation of the purity of non-spectral colors. Except as they may be explicitly defined below, all symbols are to be understood as defined in the previous paper.

By way of introduction, it is necessary to mention that usage in the definition of the purity of non-spectral colors is not uniform. This question is best clarified by considering the experimental method of determining purity. The following procedures are followed in the experimental measurement of purity. In general, for either spectral or non-spectral colors, we have a divided photometric field in one half of which appears the color of the sample while the other half is illuminated by the neutral stimulus ("white light"). Then:

(1) *For spectral colors*, a homogeneous stimulus of suitable wave length is added to the neutral stimulus and the brightnesses are adjusted until the two halves of the field are perfectly matched.

(2) *For non-spectral colors*, a homogeneous stimulus is added to *the sample field* and the brightnesses are adjusted until the two halves of the field match; i.e., by the addition of a homogeneous stimulus, the sample field is made to match the neutral.

In the measurement of purity, we deal with the several brightness components. Let them be defined as follows when the two halves of the field are matched:

$B_z \equiv$ component of brightness due to the sample.

$B_\Lambda \equiv$ component of brightness due to the homogeneous stimulus.

$B_n \equiv$ component of brightness due to the neutral stimulus.

Then, according to H. E. Ives[4] and others, the definition of purity is perfectly general for both spectral and non-spectral colors and is:

$$p \equiv \frac{B_\Lambda}{B_z} . \qquad (1)$$

As a matter of theoretical convention, Ives considers B_Λ as a positive quantity when the homogeneous stimulus is added to the neutral stimulus and as a negative quantity when it is added to the sample.[5]

[3] Formula 21, J.O.S.A. & R.S.I., *9*, p. 510.

[4] J. F. I., p. 38–39, Jan. 1923, and elsewhere. Dr. Ives has informed me in a letter of August 4, 1925 that he believes he has followed consistently in all of his papers the definition and practice ascribed to him in the present paper.

[5] The basis for this convention is the theoretical idea of operating *only* on the original neutral field to match the sample in all cases. This, of course, is actually impossible in matching non-spectral colors. To conserve the idea we may think of *subtracting* the homogeneous stimulus from the neutral field instead of adding it (as in actual practice) to the sample field.

By this convention, the purity of non-spectral colors is always *negative*. On the other hand, according to Troland[6] purity for non-spectral colors is defined as

$$p \equiv \frac{B_A}{B_n} = \frac{B_A}{B_z + B_A} . \qquad (2)$$

In what follows we shall deal only with purity as defined by Ives (1 above).

We now proceed to show how the previous formula for the purity of spectral colors (Formula 21, J.O.S.A., *9*, p. 510) is to be interpreted and applied to non-spectral colors.

The demonstration falls naturally into four parts, viz.:

I. Consideration of the form of the formula as independent of the choice of colors arbitrarily said to have unit purity.

II. Formulation of conditions fixing the reference point of unit purity so as to be consistent with the adopted definition of purity.

III. Solution for the trilinear coordinates of the reference point of unit purity.

IV. Reduction of the purity formula for purples to its most simple terms.

I

The previous derivation of "absolute purity," expressed by the numerator in the original formula, is perfectly general, applying to non-spectral as well as spectral colors. The numerator therefore remains unchanged.

We now consider the denominator. In the previous formula, R, G, and B were defined as the trilinear coordinates of the intersection of the locus of spectrum colors (arbitrarily said to have unit purity) with a straight line drawn from the center of the triangle (zero purity) through the point, r, g, b, to which the value of p given by the formula applies.[7] It is evident the formula would be unchanged in form if we should arbitrarily assign unit purity to *any* point on the above mentioned straight line and define R, G, and B as the coordinates of this point. The effect would be simply that we had changed our *standard of unit purity*.

[6] Report O. S. A. Colorim. Com., J.O.S.A. & R.S.I., *6*, pp. 527–596; 1922. The definition is not given *explicitly* in the report but is *implicit* in Fig. 9, p. 575 and Tab. 14 B, p. 585. For the explicit statement of the definition of purity as used in this report, I am indebted to Troland's colleague, Mr. E. A. Weaver. (Letter of Aug. 5, 1925, Weaver to Priest.)

[7] Fig. 1, J.O.S.A. & R.S.I., *9*, p. 507.

To consider the matter generally, let the above mentioned straight line be projected backwards into the region marked "Purples" in Fig. 1 of the previous paper, and define symbols as follows:

R_A, G_A, B_A are the coordinates of the intersection of the line with the curve marked "Spectrum." (Previously called simply R, G, B.) Let C_A be the least of these coordinates.

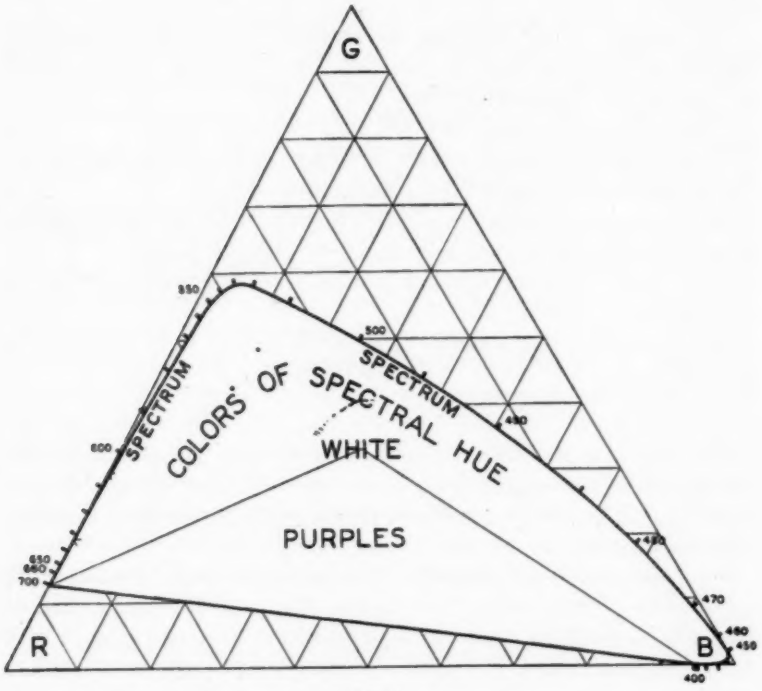

FIG. 1

(Reproduction of Fig. 1, J.O.S.A. and R.S.I., 9, p. 507)

R_c, G_c, B_c are the coordinates of another point also arbitrarily said to have unit purity but lying on the segment of the line falling in the region marked "Purples." Let C_c be the least of these coordinates. r, g, b are the coordinates of *any* point on the above mentioned line. Let c be the least of these coordinates.

If the point, r, g, b, lie between the center and the curve marked "Spectrum," the previous formula applies directly, namely:

$$p=\frac{1-\dfrac{c}{L_r r+L_g g+L_b b}}{1-\dfrac{C_A}{L_r R_A+L_g G_A+L_b B_A}}. \tag{3}$$

If the point, $r, g, b,$ lie in the region marked "Purples," we refer its purity to a purple of unit purity and, in like manner to the above, still conceiving of p as a positive quantity, we have

$$p=\frac{1-\dfrac{c}{L_r r+L_g g+L_b b}}{1-\dfrac{C_c}{L_r R_c+L_g G_c+L_b B_c}}. \tag{4}$$

II

The problem now reduces to that of fixing the reference point of unit purity so as to be consistent with Ives' definition of purity for non-spectral colors. That is, we require in the purple region the locus of points of unit purity, analogous to the "Spectrum" curve which is by definition a locus of points of unit purity. We propose to impose conditions for finding this locus so that purities expressed in accord with it shall be also in accord with our adopted definition of purity. Let the subscripts Λ and c, when attached to ρ, γ, and β (defined in the previous paper) have the same significance as when attached to R, G, and B. Let B_c be the brightness of the purple of unit purity represented by the coordinates R_c, G_c, B_c.

By the adopted general definition of purity

$$\frac{B_A}{B_c}=1.000\ .\ .\ .$$

or

$$B_A=B_c\ .$$

These brightnesses (in arbitrary units) are given respectively by

$$B_A=L_r\rho_A+L_g\gamma_A+L_b\beta_A \tag{5}$$

and

$$B_c=L_r\rho_c+L_g\gamma_c+L_b\beta_c\ . \tag{6}$$

Consequently,

$$L_r\rho_c+L_g\gamma_c+L_b\beta_c=\boldsymbol{B}_A\ . \tag{7}$$

Since R_A, G_A, B_A and R_c, G_c, B_c represent complementary colors, we have

$$\rho_A+\rho_c=\rho_n$$
$$\gamma_A+\gamma_c=\gamma_n$$
$$\beta_A+\beta_c=\beta_n$$

where ρ_n, γ_n, and β_n are the amounts of elementary stimuli for neutral. Then, since

$$\rho_n=\gamma_n=\beta_n$$

we have

$$\rho_A+\rho_c=\gamma_A+\gamma_c=\beta_A+\beta_c\ . \tag{8}$$

Finally, we have, by definition in the previous paper,

$$L_r+L_g+L_b=1.000\ .\ .\ . \tag{9}$$

III

We now proceed to find R_c, G_c, and B_c as fixed by the above conditions.

From 5 and 7,

$$L_r(\rho_A+\rho_c)+L_g(\gamma_A+\gamma_c)+L_b(\beta_A+\beta_c)=2\boldsymbol{B}_A\ . \tag{10}$$

From 8, 9, and 10,

$$\left.\begin{aligned}\rho_A+\rho_c&=2\boldsymbol{B}_A\\ \gamma_A+\gamma_c&=2\boldsymbol{B}_A\\ \beta_A+\beta_c&=2\boldsymbol{B}_A\end{aligned}\right\}\ . \tag{11}$$

From 11,

$$\left.\begin{aligned}\rho_c&=2\boldsymbol{B}_A-\rho_A\\ \gamma_c&=2\boldsymbol{B}_A-\gamma_A\\ \beta_c&=2\boldsymbol{B}_A-\beta_A\end{aligned}\right\}\ . \tag{12}$$

From 12,

$$\rho_c+\gamma_c+\beta_c=6\boldsymbol{B}_A-(\rho_A+\gamma_A+\beta_A)\ . \tag{13}$$

From 12 and 13 and the fundamental general definition of R, G, and B,

$$\left.\begin{aligned}R_c&=\frac{2\boldsymbol{B}_A-\rho_A}{6\boldsymbol{B}_A-(\rho_A+\gamma_A+\beta_A)}\\ G_c&=\frac{2\boldsymbol{B}_A-\gamma_A}{6\boldsymbol{B}_A-(\rho_A+\gamma_A+\beta_A)}\\ B_c&=\frac{2\boldsymbol{B}_A-\beta_A}{6\boldsymbol{B}_A-(\rho_A+\gamma_A+\beta_A)}\ .\end{aligned}\right\} \tag{14}$$

IV

Substitution of the above values of R_c, G_c, and B_c in (4) would lead to numerical evaluation of p. However, the formula may be greatly simplified as follows:

From 14, 9, and 5, we have

$$L_rR_c+L_gG_c+L_bB_c=\frac{B_A}{6B_A-(\rho_A+\gamma_A+\beta_A)}. \tag{15}$$

It is readily seen in the mixture triangle (Fig. 1) that

$$C_c=R_c \text{ or } G_c \text{ or } B_c$$

according as

$$c=r \text{ or } g \text{ or } b\ .$$

Suppose that[8] $c=g$. Then the denominator of formula 4 is:

$$1-\frac{G_c}{L_rR_c+L_gG_c+L_bB_c}.$$

From 14 and 15,

$$\frac{G_c}{L_rR_c+L_gG_c+L_bB_c}=\frac{\dfrac{2B_A-\gamma_A}{6B_A-(\rho_A+\gamma_A+\beta_A)}}{\dfrac{B_A}{6B_A-(\rho_A+\gamma_A+\beta_A)}}$$

$$=\frac{2B_A-\gamma_A}{B_A}$$

$$=2-\frac{\gamma_A}{B_A}. \tag{16}$$

From 16 and 5,

$$\frac{G_c}{L_rR_c+L_gG_c+L_bB_c}=2-\frac{\gamma_A}{L_r\rho_A+L_g\gamma_A+L_b\beta_A}. \tag{17}$$

Since, by general definition of R, G, and B,

$$\rho_A=R_A(\rho_A+\gamma_A+\beta_A)$$

[8] If two of the elementary stimuli correspond to the "ends of the spectrum," g is, in general, the least of the three coordinates for non-spectral colors. This is true for the elementaries chosen by the O. S. A. Colorimetry Committee. However, since this choice is not necessary, we shall consider also the cases of $c=r$ and $c=b$ in what follows.

and $$\gamma_A = G_A(\rho_A + \gamma_A + \beta_A)$$

and $$\beta_A = B_A(\rho_A + \gamma_A + \beta_A) ,$$

we have, from 17,

$$\frac{G_c}{L_r R_c + L_g G_c + L_b B_c} = 2 - \frac{\gamma_A}{(L_r R_A + L_g G_A + L_b B_A)(\rho_A + \gamma_A + \beta_A)} . \quad (18)$$

Since

$$\frac{\gamma_A}{\rho_A + \gamma_A + \beta_A} = G_A ,$$

we have, from 18,

$$\frac{G_c}{L_r R_c + L_g G_c + L_b B_c} = 2 - \frac{G_A}{L_r R_A + L_g G_A + L_b B_A} . \quad (19)$$

From 19 and 4; and the condition that $c = g$, we have

$$p = \frac{1 - \dfrac{g}{L_r r + L_g g + L_b b}}{1 - 2 + \dfrac{G_A}{L_r R_A + L_g G_A + L_b B_A}}$$

or

$$p = - \frac{1 - \dfrac{g}{L_r r + L_g g + L_b b}}{1 - \dfrac{G_A}{L_r R_A + L_g G_A + L_b B_A}} . \quad (20)$$

If $c = r$ or if $c = b$, we have in like manner either

$$p = - \frac{1 - \dfrac{r}{L_r r + L_g g + L_b b}}{1 - \dfrac{R_A}{L_r R_A + L_g G_A + L_b B_A}} \quad (21)$$

or

$$p = - \frac{1 - \dfrac{b}{L_r r + L_g g + L_b b}}{1 - \dfrac{B_A}{L_r R_A + L_g G_A + L_b B_A}} \quad (22)$$

according as $c = r$ or $c = b$.

It is readily seen in the mixture triangle (Fig. 1) that, when the point, R, G, B, and the point, r, g, b, are collinear with the center $(1/3, 1/3, 1/3)$ and lie on *opposite* sides of *it*, R or G or B is the greatest of R, G, and B according as r, g, or b is the least.

Considering all symbols as having their original meanings, we see that the formula for the purity of non-spectral colors is exactly the same as the original formula for spectral colors except that

(1) the sign is reversed; and

(2) C is the greatest instead of the least of R, G, B.

A word of comment is in order concerning the reversal of sign. If numerical values be substituted in formula 20 (or 21 or 22), p will be found to be positive in all cases. This is because the derivation above has been based on the initial concept of p as a positive quantity (*vide supra*). Indeed, if we think of purity simply as a ratio of two brightnesses, no other concept is possible for we cannot conceive of a negative brightness. Ives' convention of assigning the negative sign to the brightness of the homogeneous stimulus added to the sample may, however, be superimposed upon the foregoing derivation, and will result in negative values of p for colors of non-spectral hue.

In conclusion we have, from what has been given above, the following rule for the general application of formula 21 of the previous paper[9] to non-spectral as well as spectral colors:

All symbols are taken in their original meaning except that

$$C = R$$

or

$$C = G$$

or

$$C = B$$

according as

$$c = r$$

or

$$c = g$$

or

$$c = b\,.$$

This will result in positive values of p for spectral colors (C being least of R, G, and B), and negative values of p for non-spectral colors

[9] J.O.S.A. & R.S.I., *9*, p. 510.

(C being greatest of R, G, and B), all in accord with Ives' convention as to algebraic signs.

The routine computation of p is greatly facilitated by plots of the denominators of formulas 20, 21, and 22 (23a, 23b, and 23c of the previous paper[10]) as functions of Λ. One has only then to compute the numerator and read the denominator from the curve for

$$1-\frac{R}{L_rR+L_gG+L_bB}$$

or

$$1-\frac{G}{L_rR+L_gG+L_bB}$$

or

$$1-\frac{B}{L_rR+L_gG+L_bB}$$

according as

$$c=r$$

or

$$c=g$$

or

$$c=b\ .$$

I desire to thank Dr. F. G. Brickwedde for his criticism resulting in important changes in the manuscript of this paper.

BUREAU OF STANDARDS,
OCTOBER 1925–FEBRUARY 1926.

[10] J.O.S.A. & R.S.I., *9*, p. 511.

Corrections to previous paper:

For the three lines immediately above formula 24, J.O.S.A. & R.S.I., *9*, p. 511, substitute the following:

"Dr. Tuckerman has also derived the symmetrical formula."

In line 8, p. 512, for "24" read "23a, 23b, 23c."

THE COMPUTATION OF COLORIMETRIC PURITY*

By Deane B. Judd**

ABSTRACT

Formulas expressing colorimetric purity in terms of given trilinear coördinates have been independently derived by Ives and Priest. This paper gives (1) the derivation of still another form of purity formula; (2) a discussion of the computational methods which are, at once, accurate and speedy, together with an analysis of the various formulas with regard to convenience for routine computing; (3) the extension of the new formulas to purity for purples; and, (4) a general solution for the trilinear coördinates (r,g,b) of a color of purity, p, in terms of the coördinates (R,G,B) of the spectrum and the luminosity coefficients (L_r,L_g,L_b).

INTRODUCTION[1]

Formulas for the computation of colorimetric purity from given "trilinear coördinates" and "luminosity coefficients" have been independently derived by H. E. Ives[2] and I. G. Priest. Tuckerman has shown that the formulas of Ives and Priest are mathematical identities provided certain restrictions are placed upon the variables, although the formulas are of very different form. He has derived from Ives' formula, three formulas of which some one is always Priest's formula. The latter thus proves to be a special case of a more general formula. Tuckerman also gives the formula in still another form.[3] I have recently derived by independent argument still other formulas for colorimetric purity which also prove to be convertible into those above mentioned.[4]

* Partial report No. 6 on cooperative investigations in colorimetry by the Bureau of Standards and the Munsell Research Laboratory. Published by permission of the Director of the Bureau of Standards, U. S. Department of Commerce and the Director of the Munsell Research Laboratory.

** Munsell Research Associate in Colorimetry at the National Bureau of Standards.

[1] It is presumed that the reader is acquainted with the following previous papers particularly the last named:

1. Ives: "Transformation of Color-Mixture Equations," J. F. I., *180*, pp. 673–701; December, 1915, and J. F. I., *195*, pp. 23–44; January, 1923.

2. Troland: "Report of the Colorimetry Committee of the O. S. A. 1920–21," J.O.S.A. & R.S.I., *6*, pp. 527–596; August, 1922.

3. Priest et al: "The Computation of Colorimetric Purity," J.O.S.A. & R.S.I., *9*, pp. 503–520, November, 1924.

The notation adopted in the last named paper will be followed throughout the present discussion.

[2] See Priest, Loc. cit., p. 510.

[3] Priest, Loc. cit. Bottom of page 511, also Appendix.

[4] Formulas given by Seitz (Zs. für Sinnesphysiol., *54*, Abt. 2, p. 146; 1922) and by Martin (Colour and Methods of Colour Reproduction, p. 133) are presumably intended as purity formulas, but do *not* give purity as measured in "monochromatic analysis." Martin's formula

Following Priest's notation, the formulas in question stand as follows:

$$p=\left(\frac{r-g}{rL_r+gL_g+bL_b}\right)\left(\frac{RL_r+GL_g+BL_b}{R-G}\right) \quad \text{[Ives R-G]}$$

$$p=\left(\frac{b-r}{rL_r+gL_g+bL_b}\right)\left(\frac{RL_r+GL_g+BL_b}{B-R}\right) \quad \text{[Ives B-R]}$$

$$p=\left(\frac{b-g}{rL_r+gL_g+bL_b}\right)\left(\frac{RL_r+GL_g+BL_b}{B-G}\right) \quad \text{[Ives B-G]}$$

$$p=\frac{1-\dfrac{c}{rL_r+gL_g+bL_b}}{1-\dfrac{C}{RL_r+GL_g+BL_b}} \quad \text{[Priest]}$$

where

c = least of r, g, b

C = least of R, G, B

$$p=\frac{1-\dfrac{r}{rL_r+gL_g+bL_b}}{1-\dfrac{R}{RL_r+GL_g+BL_b}} \quad \text{[Tuckerman R]}$$

$$p=\frac{1-\dfrac{g}{rL_r+gL_g+bL_b}}{1-\dfrac{G}{RL_r+GL_g+BL_b}} \quad \text{[Tuckerman G]}$$

$$p=\frac{1-\dfrac{b}{rL_r+gL_g+bL_b}}{1-\dfrac{B}{RL_r+GL_g+BL_b}} \quad \text{[Tuckerman B]}$$

---

apparently gives what Priest has defined as "absolute purity," which is meaningful only in relation to the particular elementary stimuli the system is referred to. (Priest: Loc. cit. pp. 508–509.) Semi-geometric formulas developed by Froelich (J.O.S.A. & R.S.I., *9*, No. 1, pp. 37–41; 1924) and Guild (Trans. Opt. So., *26*, No. 3, p. 162; 1924–25) do give the purity of monochromatic analysis, but involve a term (α-Froelich, ρ Guild) which must be evaluated graphically by measuring two distances on the color triangle; hence these formulas in their published form are not comparable to the purely algebraic formulas. By reducing this term to its analytic equivalent, both Froelich's and Guild's formulas fall readily into identity with any of the forms recorded in the present paper.

$$p=\frac{3-\dfrac{1}{rL_r+gL_g+bL_b}}{3-\dfrac{1}{RL_r+GL_g+BL_b}} \quad \text{[Tuckerman, Symmetrical]}$$

$$p=\frac{RL_r+GL_g+BL_b}{RL_r+GL_g+BL_b+\dfrac{r-R}{1-3r}} \quad \text{[Judd R]}$$

$$p=\frac{RL_r+GL_g+BL_b}{RL_r+GL_g+BL_b+\dfrac{g-G}{1-3g}} \quad \text{[Judd G]}$$

$$p=\frac{RL_r+GL_g+BL_b}{RL_r+GL_g+BL_b+\dfrac{b-B}{1-3b}} \quad \text{[Judd B]}$$

$$p=\frac{rL_r+gL_g+bL_b-\dfrac{r-R}{1-3R}}{rL_r+gL_g+bL_b} \quad \text{[Judd } r\text{]}$$

$$p=\frac{rL_r+gL_g+bL_b-\dfrac{g-G}{1-3G}}{rL_r+gL_g+bL_b} \quad \text{[Judd } g\text{]}$$

$$p=\frac{rL_r+gL_g+bL_b-\dfrac{b-B}{1-3B}}{rL_r+gL_g+bL_b} \quad \text{[Judd } b\text{]}$$

In what follows, these several formulas will be briefly referred to by the designations appended to them in brackets in the above list. The conditions which must be met in order that these various forms be true identities are as follows[5]:

$$(1) \qquad R+G+B=1\ ,$$

$$(2) \qquad r+g+b=1\ ,$$

$$(3) \qquad L_r+L_g+L_b=1\ ,$$

$$(4) \qquad \frac{1-3r}{1-3G}=\frac{1-3g}{1-3R}=\frac{1-3b}{1-3B}\ .$$

[5] For the identity of some forms all these conditions are not necessary. However, if all the conditions are satisfied, all the forms are identical.

It is apparent that there is considerable latitude for choice in selecting a particular formula for numerical computation. The purposes of this paper are:

1. To give the derivation of the new formulas.
2. To investigate the formulas from the point of view of convenience and accuracy in numerical computation.
3. To develop and formulate the rules of numerical computation which will insure the smallest possible uncertainty in the computed value of p.[6]
4. To show how the formulas are applied to colors of non-spectral hue (purples) and to discuss the definition of purity for purples.
5. To give a solution for the trilinear coördinates (r, g, b) for any point of given purity (p) in terms of the spectrum scale coördinates (R, G, B).

I. THE DERIVATION OF THE NEW FORMULA

If ρ_λ, γ_λ, β_λ be the respective amounts of the elementary stimuli required to result in the same response as that to homogeneous radiation of wave length (λ), then the relations:

$$R=\frac{\rho_\lambda}{\rho_\lambda+\gamma_\lambda+\beta_\lambda}\ ; \qquad G=\frac{\gamma_\lambda}{\rho_\lambda+\gamma_\lambda+\beta_\lambda}\ ; \qquad B=\frac{\beta_\lambda}{\rho_\lambda+\gamma_\lambda+\beta_\lambda}\ .$$

define R, G, and B. Should neutral heterogeneous radiation be added to the stimulus, each elementary magnitude would be increased by an amount, ω, and the relations:

$$r=\frac{\rho_\lambda+\omega}{\rho_\lambda+\gamma_\lambda+\beta_\lambda+3\omega}\ ; \qquad g=\frac{\gamma_\lambda+\omega}{\rho_\lambda+\gamma_\lambda+\beta_\lambda+3\omega}\ ; \qquad b=\frac{\beta_\lambda+\omega}{\rho_\lambda+\gamma_\lambda+\beta_\lambda+3\omega}\ .$$

define r, g, and b.

The purity of this mixture is defined as:

$$p=\boldsymbol{B}_\lambda/(\boldsymbol{B}_\lambda+\boldsymbol{B}_w)\ , \qquad \boldsymbol{B}_\lambda+\boldsymbol{B}_w=\boldsymbol{B}_m=\boldsymbol{B}_e\ ;$$

[6]Priest has already pointed out certain practical precautions in the numerical application of these formulas (J.O.S.A. & R.S.I., 9, pp. 512, 516, and 518), but has not treated the question fully. It may be inferred from his paper that of the three Tuckerman forms (Tuckerman R, Tuckerman G, Tuckerman B) the special case designated as Priest's formula is the most accurate. (J.O.S.A. & R.S.I., 9, pp. 516–519.) However, no general proof is given nor are the underlying reasons for the numerical discrepancies in his computed results discussed fully. The present study of the formulas has been undertaken at Mr. Priest's request, and much of the material herein developed is the direct result of his suggestions, and of those of Dr. Tuckerman.

where:

B_λ = brightness of the spectral component of the mixture field,
B_w = brightness of the white component of the mixture field.
B_m = brightness of the mixture field,
B_s = brightness of the sample field.

But B_λ can be expressed in terms of the elementary stimuli (ρ_λ, γ_λ, β_λ) and their respective luminosity coefficients (L_r, L_g, L_b):

$$B_\lambda = \rho_\lambda L_r + \gamma_\lambda L_g + \beta_\lambda L_b .$$

In the same terms:

$$B_m = (\rho_\lambda + \omega)L_r + (\gamma_\lambda + \omega)L_g + (\beta_\lambda + \omega)L_b = \omega(L_r + L_g + L_b) + \rho_\lambda L_r + \gamma_\lambda L_g + \beta_\lambda L_b .$$

$= \omega + B_\lambda$; whence : $B_w = \omega$.

Then, substituting:

$$p = \frac{B_\lambda}{B_\lambda + B_w} = \frac{\rho_\lambda L_r + \gamma_\lambda L_g + \beta_\lambda L_b}{\rho_\lambda L_r + \gamma_\lambda L_g + \beta_\lambda L_b + \omega} .$$

From the definitions of R, G, and B, the purity may be expressed:

$$p = \frac{RL_r + GL_g + BL_b}{RL_r + GL_g + BL_b + \omega/(\rho_\lambda + \gamma_\lambda + \beta_\lambda)} .$$

Also from the definitions of r, g, b, R, G, and B:

$$\frac{\omega}{\rho_\lambda + \gamma_\lambda + \beta_\lambda} = \frac{r - R}{1 - 3r} = \frac{g - G}{1 - 3g} = \frac{b - B}{1 - 3b} .$$

The formula may be thrown into the form of the definition for the purity of colors of spectral hue by virtue of the fact that $\rho_\lambda + \gamma_\lambda + \beta_\lambda$ is constant for a given wave length, since ρ_λ, γ_λ and β_λ are separately fixed for a given wave length, regardless of the purity, as the subscripts indicate. Thus:

$$p = b_\lambda/(b_\lambda + b_w),$$

where:

$$b_\lambda = B_\lambda/(\rho_\lambda + \gamma_\lambda + \beta_\lambda),$$

= brightness of the spectral component of the mixture field in arbitrary units which change as the wave length changes, and

$$b_w = B_w/(\rho_\lambda + \gamma_\lambda + \beta_\lambda),$$

= brightness of the white component of the mixture field in the same units.

From the definitions of r, g, b, R, G, and B:

$$b_\lambda = RL_r + GL_g + BL_b ,$$

$$b_w = \frac{r-R}{1-3r} = \frac{g-G}{1-3g} = \frac{b-B}{1-3b} .$$

Substituting these values of b_λ and b_w in the relation: $p = b_\lambda/(b_\lambda + b_w)$, gives the forms designated in the summary (see Introduction) as Judd R, Judd G, and Judd B.

II. COMPUTATIONAL ACCURACY AND CONVENIENCE OF THE FORMULAS

The formulas for the computation of colorimetric purity contain the following quantities:

1. L_r, L_g, and L_b—the chromatic luminosity coefficients, constant for all wave lengths and all purities, and regarded as evaluated once for all by general colorimetric investigation.

2. R, G, and B—the trilinear coördinates of the spectrum, functions of the wave length but not of the purity, and read for each particular case from an interpolation graph.

3. r, g, and b—the trilinear coördinates of the point whose purity (p) it is desired to evaluate. These may be regarded as accurately given. The problem to be solved by any of the formulas is hypothetical, thus: If r, g, and b are the trilinear coördinates of a color in a system whose "luminosity coefficients" and "spectrum scale" for each even 10 mμ are known within the limits of experimental accuracy, what is the purity of that color?

In any particular problem, then, the accuracy of the result depends on the accuracy of the evaluation of (R, G, B) from the spectrum scale. If R, G, and B are correctly determined from the interpolation graph they will meet two conditions, already stated (Page 135):

$$(1) \qquad R+G+B = 1$$

$$(4) \qquad \frac{1-3r}{1-3R} = \frac{1-3g}{1-3G} = \frac{1-3b}{1-3B} .$$ [7]

Conditions (2) and (3) of the same group are easily complied with by carrying the computation of the quantities involved out to one more place (say four) than is to be carried through the subsequent calculation,

[7] This condition (4) will be recognized as that which makes the point (r, g, b) and the point (R, G, B) collinear with the neutral or white point (1/3, 1/3, 1/3). It is the analytical expression for the necessity that the color represented by the point (r, g, b) have the same dominant wave length (λ) as the spectral color represented by the coordinates (R, G, B). Condition (4) will be hereinafter referred to as the collinearity condition.

and then cutting back to three places so that the sum is exactly unity. If R, G, and B are so chosen as to satisfy the conditions (1) and (4), then, all forms of the purity formula are identical and will yield identical results. If there be an error in the determination of R, G, or B so that they do not meet these conditions, none of the forms of purity formula yield the true purity. If, however, for a given error in R, G, or B, one form gives a purity in error by, say, 0.10 (that errors as large as this can occur may be seen from reference to Column 17, Table 2, Appendix) and another, a purity in error by 0.001, the latter form may be said to be computationally accurate in comparison to the former. It is a matter of importance in colorimetric calculation to know which forms are computationally precise, and which are not.

The largeness of the errors in purity for some cases arises from the fact that in all of the formulas differences involving R, G, and B occur.

TABLE 1. *The Convenience of the Purity Formulas*

Designation of Purity Formula	Criterion of Inconvenience	Wave length Region of inconvenience from O. S. A. Elementaries
Ives $R-G$	When $r-g\rightarrow 0$	560 mμ to 580 mμ
Ives $B-R$	When $b-r\rightarrow 0$	500 mμ to 510 mμ
Ives $B-G$	When $b-g\rightarrow 0$	490 mμ to 495 mμ
Tuckerman R	When $^{*}RL_r+GL_g+BL_b\rightarrow R$	560 mμ to 580 mμ
Tuckerman G	When $RL_r+GL_g+BL_b\rightarrow G$	560 mμ to 584 mμ *and* 440 mμ to 466 mμ
Tuckerman B	When $RL_r+GL_g+BL_b\rightarrow B$	494 mμ to 497 mμ
Priest	When$^{**}RL_r+GL_g+BL_b\rightarrow C$	None
Judd R or r	When $1-3r\rightarrow 0$	520 mμ to 540 mμ
Judd G or g	When $1-3g\rightarrow 0$	485 mμ to 490 mμ *and* 585 mμ to 600 mμ
Judd B or b	When $1-3b\rightarrow 0$	495 mμ to 498 mμ
Tuckerman Symmetrical	When $RL_r+GL_g+BL_b\rightarrow 1/3$	488 mμ to 502 mμ *and* 600 mμ to 720 mμ

* To compare these quantities as the wave length varies, refer to Fig. 1. This shows R, G, B, and $RL_r+GL_g+BL_b$ as functions of the wave length, the excitations and luminosity coefficients involved being those recommended by the O. S. A. Committee on Colorimetry. (Cf. Troland: Loc. cit., 548–551.)

** C is the least of R, G, and B. A glance at Fig. 1 will show that this definition is equivalent to $C=R$, $\lambda<504$ mμ, and $C=B$, $\lambda>504$ mμ. It becomes evident, also, that C never approaches $RL_r+GL_g+BL_b$; hence there is no region of inconvenience.

Whenever one of these differences becomes small in comparison to the values of R, G, and B, the error in purity will be much larger than the uncertainty inherent in R, G, and B.

A glance at the formulas themselves will show the conditions under which they, severally, lose computational precision. Each one of these conditions defines a wave length region (dependent somewhat on the choice of elementaries) in which the formula becomes impractical for computation. These wave length regions approximately evaluated by

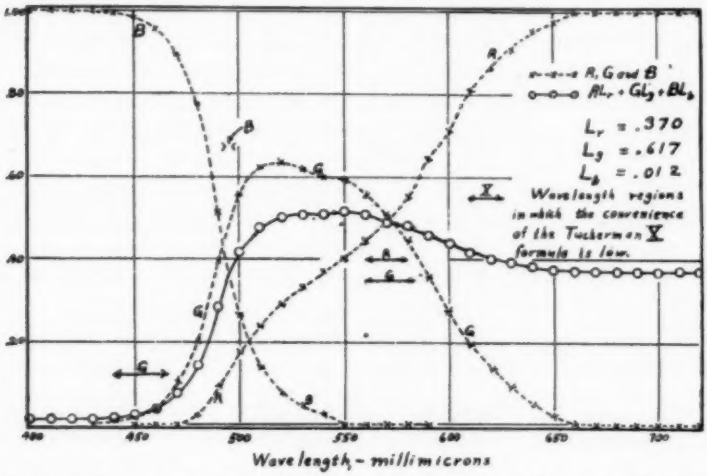
R, G and B
$RL_r + GL_g + BL_b$
$L_r = .370$
$L_g = .617$
$L_b = .012$
X
Wavelength regions in which the convenience of the Tuckerman X formula is low.
Wave length - millimicrons

FIG. 1.—*The spectrum coordinates* (R, G, B) *and* $\mathbf{b}_\lambda$ *as functions of the wave length. The wave length regions of inconvenience for the Tuckerman X formulas* (X *being either* R, G, *or* B) *are also indicated.*

reference to the O.S.A.[8] excitations and luminosity coefficients are given in Table 1.

A reference to Table 1 shows that all forms of purity formula save one become inconvenient for practical use in at least one wave length region. The special form of Tuckerman's formula designated by *Priest* so chooses among Tuckerman R, Tuckerman G, and Tuckerman B as to avoid the wave length regions of low computational precision.

While the Tuckerman Symmetrical form has a distinct advantage in numerical computation (to be fully discussed later) it is not applicable

[8] Troland, Loc. cit., pp. 548–551, p. 575.

to the actual computation of all cases of purity as the large region of inconvenience shows.[9].

III. PROCEDURE NECESSARY FOR ACCURATE COMPUTATION

As shown in the preceding section, the only source of error that is troublesome in the solution of our hypothetical problem is the evaluation of R, G, and B. Though these three quantities may be read separately from an interpolation graph (an accurate plot of Fig. 1), still there are two conditions (1 and 4, page 135) which involve them. Thus, if any one of R, G, or B be read from the graph, the other two could be calculated from the conditions themselves. To render as harmless as possible, then, the unavoidable error in R, G, and B; two principles of selection should be followed:

1. Select that spectrum coördinate (R, G, or B) as basis for the calculation of the other two, which can be determined with the least uncertainty.

2. Select some form of purity formula whose "region of inconvenience" does not include the point (r, g, b).

The general method of purity computation embodying these two precautionary selections is:

1. Find the dominant wave length corresponding to the point (r, g, b) from a plot of the spectrum on the color triangle[10] or, better yet, by an accurate method of interpolation, such as a plot of $(R-1/3)/(B-1/3)$ against the wave length.

2. Read from an interpolation graph (an accurate plot of Fig. 1) the value of one of R, G, and B, as follows:

[9] This checks a statement made by Priest (Loc. cit., p. 512) based on actual trial computations. If we set $L_r=L_g=L_b=1/3$, and substitute in the Tuckerman Symmetrical form, p becomes indeterminate. The usefulness of the formula is, thus, seen to depend on how far the luminosity coefficients deviate from 1/3. The fact that the luminosity coefficients which have been used for the O. S. A. curves by Priest involve an L_r which is close to 1/3 (0.370) accounts for the inconvenience of the formula in the red region. It is readily seen that the Tuckerman Symmetrical form might easily be as convenient as any of the other forms provided only the luminosity coefficients are all appreciably different from 1/3. If, indeed, the luminosity coefficients, $L_r=0.45$, $L_g=0.54$, and $L_b=0.01$ be used for the O. S. A. excitations (Judd, Chromatic Visibility Coefficients by the Method of Least Squares, J.O.S.A. & R.S.I., *10*, No. 6, p. 646, June, 1925,) which fit them better to the visibility than the values used by Priest, the Tuckerman Symmetrical form could be employed in actual computation in a wave length region of extent comparable to that of any other single form of purity formula.

[10] Troland: Loc. cit. p. 575.

For $\lambda = 380$ mμ to 510 mμ, read R.
For $\lambda = 510$ mμ to 530 mμ, read G.
For $\lambda = 530$ mμ to 720 mμ, read B.[11]

3. Calculate the other two coördinates of R, G, B from the equations of condition:

$$(1) \qquad R+G+B=1$$

$$(2) \qquad \frac{1-3r}{1-3R} = \frac{1-3g}{1-3G} = \frac{1-3b}{1-3B}.$$

After this step has been completed, numerical values will be available for r, g, b; R, G, B; and L_r, L_g, L_b, which satisfy all four equations of condition.

4. Substitute these values in any of the forms of the purity formula, and compute p. Since we have complied with the conditions which make all these forms identical, the results will be identical regardless of the form used. However, it will be wise to follow the second principle of selection, and choose a convenient form of purity formula, if the computing of R, G, B (Step 3) to a large number of decimals is to be avoided.

Two alternative methods of computing purity are worthy of mention at this point because they avoid the tedious computation of R, G, B (Step 3) and still yield reliable values for purity by automatically conforming to the two principles of selection as well as to the equations of condition.

[11] The basis underlying this division on the wave length scale is merely the first principle of selection. A reference to Fig. 1 will show that the above choice of spectral coördinate avoids those whose slope with wave length change is highest and whose interpolation error therefore greatest. Since Fig. 1 refers to the O. S. A. excitations, the above division on the wave length scale should be followed only when the choice of color primaries is close to those embodied in the O. S. A. excitations. For other choices, the points of division will be somewhat shifted.

Another justification of this division of wave lengths, scarcely less important, is as follows: The intersection of the hue line defined by (r, g, b) with the spectrum line is being sought. We plan to alter the spectral coordinates (R, G, B) slightly by imposing the collinearity condition (See next step). However, to reduce the purity error to a minimum, we must so alter R, G, B, as to move the point in the general direction of the spectrum line at that wave length. Thus, for long wave lengths, hold B constant as read, and, of necessity, the change by imposing the collinearity condition must consist of varying the R–G ratio, a change which moves the point approximately along the spectrum line. This division of the wave length scale, apparently quite by chance, thus, not only effects a start from the best determined spectral coördinate, but also insures that the change for perfect collinearity does not move the point (R, G, B) from the spectrum line.

A. *Priest's Method.* The procedure in use at the National Bureau of Standards for the calculation of dominant wave length and purity is as follows[12]:

1. From the data for r, g, b the ratio $(r-1/3)/(b-1/3)$ is calculated.

2. The dominant wave length is found by applying this ratio $(r-1/3)/(b-1/3)$, or the ratio $(b-1/3)/(r-1/3)$, to a graph showing the dominant wave length as a function of these two ratios. This graph, itself, was previously prepared from the trilinear coördinates of the spectrum by plotting $(R-1/3)/(B-1/3)$ and $(B-1/3)/(R-1/3)$ against dominant wave length.

3. The whole denominator of Priest's formula, $1-C/(RL_r+GL_g+BL_b)$, corresponding to this dominant wave length is read off a graph showing the denominator as a function of the dominant wave length. (Priest's method in this form does not apply to colors of non-spectral hue.)

4. The numerator is evaluated by inserting the given data, r, g, b, in the form: $1-c/(rL_r+gL_g+bL_b)$, where c is the least of r, g, b; and the numerator is divided by the denominator to find the purity.

By the preparation of the graph referred to in Step 3, the tedious computation of R, G, B, is, in effect, accomplished once for all. Equations of condition (2) and (3) may be easily satisfied as described in Section II. A failure to meet requirements of condition (1) or (4) would appear as an error in plotting or reading the graph of Step 3. However, the graph was so carefully constructed (points plotted every 2 mμ in doubtful regions) as to make the interpolation of the denominator accurate to 0.001 of its value.

It is interesting to note that this method also complies with the two principles of selection almost perfectly. The second principle—the avoidance of "inconvenient wave length regions"—is exactly followed by the choice of the "least of the coördinates." (See Section II, footnote to Table 1.) If the first principle of selection is to be followed exactly, the division of the wave length scale of Step 2, General Method of Computation, must be met by the choice of the "least of the coördinates." A reference to Fig. 1 will show that the region 510 mμ to 530 mμ is the only one in which this is not true. For values of r, g, b which fall in this narrow wave length region, then, the value of purity is not quite

[12] There is no previous published account of the exact procedure followed in routine computation. Mr. Priest states that it is partly (Step 3) due to suggestion of Mr. C. D. Hillman of Feuffel and Esser and requests that this be acknowledged.

so certain by this method as by the rigorous, general method. However, it is still accurate to within 0.001 of its value, since the graph of the denominator may be read to that accuracy. This is several times as accurate as the experimental determination of purity for that wave length region.

B. *Purity by the Judd formulas.* The justification for proposing a new set of formulas for colorimetric purity—an entity that seems already to have adequate functional representation—is two-fold:

1. The derivation is straight-forward and yields a formula easily understandable in terms of the definition of purity.

2. The formulas involve the mixture cöordinates (r, g, b) singly (Judd R, Judd G, and Judd B) or the spectral cöordinates (R, G, B) singly (Judd r, Judd g, Judd b). This property, which is distinctive of no other form, makes possible certain shortcuts in computation, as will appear later.

The steps in the procedure which, like Priest's method, avoids the tedious computation of R, G, and B, are as follows:

1. From r, g, and b find the dominant wave length by the use of an accurate interpolation method (see Step 1, Priest's Method).

2. Read from an interpolation graph (an accurate plot of Fig. 1) $RL_r+GL_g+BL_b$, and the value of one of R, G, or B, corresponding to the dominant wave length, as follows:

For $\Lambda = 380$ mμ to 510 mμ, read R.
For $\Lambda = 510$ mμ to 530 mμ, read G.
For $\Lambda = 530$ mμ to 720 mμ, read B.[13]

3. Calculate the purity from:

$$p = \boldsymbol{b}_\Lambda/(\boldsymbol{b}_\Lambda+\boldsymbol{b}_w),$$

where $\boldsymbol{b}_\Lambda$ is the value of $RL_r+GL_g+BL_b$ read from the graph. $\boldsymbol{b}_w$ is figured from $(r-R)/(1-3r)$ or $(g-G)/(1-3g)$ or $(b-B)/(1-3b)$ according as R, G, or B is available from Step 2. The formulas made use of are Judd R, Judd G, or Judd B.

It is readily seen that the first principle of selection (for R, G, or B) is satisfied because of the wave length division of Step 2. The second principle of selection which sidesteps the regions of inconvenience is also followed by this division of the wave length scale as a reference to Table 1 will show.

[13] This division of the wave length scale is the same as that in Step 2, General Method. For colors of non-spectral hue, take Λ as the complementary wave length.

As in Priest's Method, however, there is still a narrow wave length region where the possibility of error due to lack of computational precision must be avoided by extreme care in the drafting of interpolation graphs. This region is 470 mμ to 500 mμ (see Fig. 1) where $RL_r+GL_g+BL_b$ changes rapidly with wave length. This change makes interpolation correspondingly less certain.

It is conceivable that an occasion might arise which would demand the accurate evaluation of a few purities but which did not justify the labor involved in the careful preparation of an interpolation graph (as in Priest's method or the method just described). For this situation, the following modification of the general method is proposed:

1. Steps 1 and 2 are the same. When these are completed, numerical values should be available for r,g,b; L_r, L_g, L_b; and one of R, G, and B.

2. Step 3, the tedious calculation of R, G, and B from the collinearity condition, is omitted.

3. Calculate the purity (p) from Judd r, Judd g, or Judd b, according as R, G, or B is available. These forms of purity formula are easily derived by a method similar to that used for Judd R, Judd G, and Judd B, and may be written in the following form:

$$p=(b_M-b_W)/b_M,$$

where:

$$b_M=rL_r+gL_g+bL_b$$
$$b_W=(r-R)/(1-3R)$$
$$\text{or } =(g-G)/(1-3G)$$
$$\text{or } =(b-B)/(1-3B)$$

It is to be noted that this form of purity formula involves the R, G, and B singly, a property which distinguishes it from all other forms and which is the basis of its claim to greater convenience. On this account the collinearity condition is automatically imposed by the choice of R, G, or B in Step 2 according to the division of the wave length scale. This same division of the wave length scale satisfies the two principles of selection as before. It has, therefore, perfect reliability for all wave lengths but is not particularly rapid in application because $rL_r+gL_g+bL_b$ must be evaluated separately for each purity determination. The exact extent of the unwieldiness is shown in the computation step analysis which follows.

The several procedures here outlined for computing colorimetric purity yield values of the purity which will correspond to the actual

results of monochromatic analysis by a given observer only, of course, in so far as the "color excitations" and "luminosity coefficients" used in computation actually correspond to the color vision of that observer. The problem dealt with here is merely that of converting data from the trichromatic form to the terms of homo-hetero analysis, assuming that the fundamental data are accurate.[14]

Although all of the several formulas may be applied in such a way as to yield reliable values for purity, they are not equally convenient for routine computation. We shall here compare the formulas as to simplicity in numerical application assuming that:

1. r, g, and b are given;
2. Λ has been found;
3. Terms and factors which are merely functions of Λ may be found from graphs already prepared in the simplest form.

Listing the operations then required for numerical computation of p in each case we find the following:

	Tuckerman Symmetrical	Priest	Ives	Judd R, G, B	Judd r, g, b
References to graphs.....	1	1	1	2	1
Products found by mere mental operation (only one figure in one of the factors).....	0	0	0	1	1
Products or quotients requiring slide-rule or other assistance.....	5	5	5	2	5
Differences, or sums of two terms.....	1	1	1	3	3
Sums of three terms.....	1	1	1	0	1
Total operations.....	8	8	8	8	11

[14] The best data extant are to be found as follows:
Troland: J.O.S.A. & R.S.I., *6*, pp. 527–596; August 1922;
Ives: J. F. I., *195*, pp. 23–44; January 1923;
Gibson-Tyndall: Scientific Paper of the Bureau of Standards, *19*, No. 475; August 1923;
Judd: J.O.S.A. & R.S.I., *10*, p. 647; June 1925;
Gibson: J.O.S.A. & R.S.I., *10*, p. 230; February 1925.
Redeterminations of these data are now in course at the Bureau of Standards.

The disadvantage of Judd r, g, b for routine computation is seen at the outset by the three added steps it involves. For the rest, it is apparent that with regard to mere number of separate steps the formulas are on the same footing. The Tuckerman, Priest and Judd R, G, B forms have a slight advantage over the Ives forms in that one of the differences to be found involves only an integer (1 or 3) of one digit as minuend, which makes the mental operation somewhat easier.

The Tuckerman Symmetrical form has the additional slight advantage that one of the divisions is merely the taking of a reciprocal. It is probably the most simple in numerical application where it may be safely used, but as shown in Table 1 it has the larger wave length regions of inconvenience.

The choice thus lies between the Priest formula (either Tuckerman. R, G, or B) and the new formula here developed (Judd R, G, or B) Checking off similar operations of equal difficulty in these two forms we have left:

Operations in Priest's formula not required in Judd's	*Operations in Judd's formula not required in Priest's*
3 slide-rule products	One mental product
1 sum of three terms	2 sums of 2 terms each
	1 reference to a graph

It is seen that there is little to choose between the two formulas on the basis of simplicity in numerical work. The fact that the Judd form may be applied to purity for purples unchanged, while Priest's formula in its original published form cannot, might seem to throw an advantage to the Judd formulas. However, if Priest's formulation (see Introduction) be changed to read C is R, G, or B according as c is r, g, or b, it is then applicable to colors of purple hue also.[15] Hence this advantage is more imaginary than real. Whether the Judd form or the Priest form shall be selected for routine computation seems at this time to be largely a matter of personal inclination.

IV. EXTENSION OF PURITY FORMULAS TO NON-SPECTRAL COLORS

The justification for the concept of purity lies in the experimental fact that many colors (all except purples) can be matched by mixing spectral radiation with neutral radiation.

The relation between the field and component brightnesses is:

$$B_A + B_w = B_m = B_s,$$

[15] Cf. Priest's new paper, J. O. S. A. & R. S. I. **13**, pp. 123-132; Aug. 1926.

where:

B_A = brightness of the spectral component of the mixture field,

B_w = brightness of the white component of the mixture field,

B_m = brightness of the mixture field.

B_s = brightness of the sample field.

If the concept of purity is to be extended to colors of non-spectral hue (purples) it is evident that its experimental basis will be different from that for colors of spectral hue. This basis is found in the experimental fact that for every non-spectral stimulus there can be found spectral radiation which, when mixed additively with the non-spectral stimulus, will be a color match for the neutral stimulus; again the relation between the field and component brightnesses may be written, thus

$$B_A + B_s = B_m = B_w.$$

It will be noted that in experimentally measuring the purity pertaining to a color of spectral hue, the homogeneous radiation is added to the neutral field:

$$B_A + B_w = B_s.$$

However, when a stimulus yielding a color of non-spectral hue is being considered, the homogeneous radiation is added to the other side—the sample field:

$$B_A + B_s = B_w.$$

These two experimental procedures may be made formally equivalent if we agree to give B_A a negative sign whenever it refers to homogeneous radiation added to the sample field to produce white. If p is defined as B_A/B_s for the colors of purple hue as well as for those of spectral hue the values for purity will be negative whenever the color referred to is designated by its complementary. The formal equivalence gained by this convention makes all the formulas that have been derived to apply to colors of spectral hue hold for colors of purple hue also. (See Appendix.) This is the convention adopted in all of Ives' work and will be referred to hereafter as Ives'[16] definition of purity. The perfect formal continuity yielded by this definition has gained preference for it in Great Britain.[17]

[16] Ives: Loc. cit., 1923, p. 39, also letter Ives to Priest, July 1925.

[17] J. Guild: The Geometrical Solution of Colour Mixture Problems, Trans. Opt. Soc., *26*, No. 3, p. 161; 1924–25.

The convention used to represent the purples of monochromatic analysis in the Report of the Committee on Colorimetry[18] amounts to a redefining of purity for colors of purple hue:

$$p = B_\Lambda / B_m = \begin{cases} B_\Lambda / B_s, \text{ for colors of spectral hue,} \\ B_\Lambda / B_w, \text{ for colors of non-spectral hue.} \end{cases}$$

This convention will be hereafter referred to as Troland's definition of purity, B being regarded as always positive. By the use of this definition all existing colors may be represented by purities between zero and plus one, but a separate notation must be made to distinguish spectral from non-spectral hues. Troland does this by adding "C" as suffix to the dominant wave lengths that are complementary to that of the sample field. (E.g., $\Lambda = 520$ mμ, $p = 0.50$, a green, is distinguished from one of its complementaries by $\Lambda = 520\,C$ mμ, $p = 0.50$. According to Ives' definition, these two colors would be: $\Lambda = 520$ mμ, $p = 0.50$, and $\Lambda = 520$ mμ, $p = -1.00$.)

The relation between these two conventional ways of representing colorimetric purity may be easily obtained from the foregoing definitions, thus:

$$p_T = \begin{cases} p_I, \text{ for colors of spectral hue,} \\ p_I / (p_I - 1), \text{ for colors of non-spectral hue.} \end{cases}$$

where:

p_T = purity by Troland's definition,
p_I = purity by Ives' definition.

By Ives' definition for purity, the formula becomes:

$$p = b_\Lambda / (b_\Lambda + b_w),$$

where:

$$b_\Lambda = RL_r + GL_g + BL_b$$
$$b_w = (r - R)/(1 - 3r)$$
$$\text{or } = (g - G)/(1 - 3g)$$
$$\text{or } = (b - B)/(1 - 3b)$$

This formula is general for all colors, the negative purities for the colors of purple hue appear because of the algebraic sign and magnitude of b_w.[19]

[18] L. T. Troland: Loc. cit., p. 575, 583. Also letter Weaver to Priest, August 1925.

[19] b_Λ and b_w are equal to $B_\Lambda/(\rho_\lambda + \gamma_\lambda + \beta_\lambda)$ and $B_w/(\rho_\lambda + \gamma_\lambda + \beta_\lambda)$ only when p refers to colors of spectral hue. The difficulty of attaching any brightness significance to b_Λ and b_w as defined in this section when p refers to purples is readily seen from equations 3 and 6 of the Appendix. It seems preferable, therefore, to regard b_Λ and b_w merely as abbreviations for functions which may be identified as brightness components only when the homogeneous radiation is added to the neutral field.

The formula referring to colors of spectral hue is recognized in each case as identical with that already derived. For formal proof of the formulas given for the colors of purple hue, reference must be had to the Appendix.

V. THE COÖRDINATES (r, g, b) OF THE LINES OF EQUAL PURITY, (p) IN TERMS OF THE COÖRDINATES OF THE SPECTRUM (R, G, B)

The inscription of the lines of equal purity in the color triangle has heretofore been accomplished apparently by graphical methods, making use of the observed visibility[20] which has been regarded as better determined than the visibility dependent on the function $\rho_\lambda L_r + \gamma_\lambda L_g + \beta_\lambda L_b$. It has been shown by the author,[21] however, that with properly chosen luminosity coefficients, the visibility calculated from the color mixture curves can be made to agree with the adopted visibility function to the same degree of approximation as do the experimental means for large numbers of observers from separate set-ups.[22] In view of this agreement, it has been deemed worth while to work out the coördinates of any color in terms of the purity, the luminosity coefficients, and the coördinates of the spectrum scale. The derivation of the formula follows:

Take the Judd R formula for purity, and rewrite:

$$p = \frac{b_A}{b_A + (r-R)/(1-3r)},$$

where:

$$b_A = RL_r + GL_g + BL_b$$

r may be solved for in terms of p, b_A, and R. This solution yields:

$$r = \frac{b_A(1-p)/p + R}{3b_A(1-p)/p + 1}.$$

Similarly from Judd G and Judd B, are obtained:

$$g = \frac{b_A(1-p)/p + G}{3b_A(1-p)/p + 1}; \text{ and } b = \frac{b_A(1-p)/p + B}{3b_A(1-p)/p + 1}.$$

If we set $p = 1$, we shall find that $r = R$, $g = G$, and $b = B$. This fact may be interpreted to mean that R, G, B are the coördinates of the point

[20] Froelich, Clara L., Algebraic Methods for the Calculation of Color Mixture Transformation Diagrams, J.O.S.A. & R.S.I., *9*, No. 1, pp. 37–41; July 1924.

[21] Judd: Loc. cit., p. 649.

[22] Gibson-Tyndall: Loc. cit., p. 174.

whose purity is unity. The spectral colors have been arbitrarily set upon as of unit purity, and following out this definition (R, G, B) is the point where the line joining (r, g, b) and the center $(1/3, 1/3, 1/3)$ cuts the spectrum line. However, if we interpret R, G, B as the coordinates of the point whose purity is unity, the formulas hold whether that point be defined as on the spectrum line or not.

Since these formulas were derived from the purity formulas, the restrictions there placed on the independent variables, namely:

$$(1) \qquad R+G+B=1\ ,$$

$$(3) \qquad L_r+L_g+L_b=1\ .$$

must be imposed here also. If this is done, the r, g, b found by substitution will necessarily satisfy the two conditions:

$$(2) \qquad r+g+b=1\ ,$$

$$(4) \qquad \frac{1-3r}{1-3R}=\frac{1-3g}{1-3G}=\frac{1-3b}{1-3B}\ .$$

If p is set less than zero and a line inscribed in the triangle from the coördinates by the above formulas it will be found to course through the region of colors of purple hue, and, of course, to yield a purity agreeing with that by Ives' definition.

If it is desired to inscribe the line of equal purity in the region of purple colors according to Troland's definition, r, g, and b may be solved for from:

$$p=\boldsymbol{b}_A(1-3r)/(R-r);$$
$$p=\boldsymbol{b}_A(1-3g)/(G-g);$$
$$p=\boldsymbol{b}_A(1-3b)/(B-b).$$

This yields:

$$r=(\mathrm{R}-\boldsymbol{b}_A/p)/(1-3\boldsymbol{b}_A/p);$$
$$g=(G-\boldsymbol{b}_A/p)/(1-3\boldsymbol{b}_A/p);$$
$$b=(\mathrm{B}-\boldsymbol{b}_A/p)/(1-3\boldsymbol{b}_A/p).$$

The purity line thus obtained is the same as before but differently numbered. The result may also be obtained from the previous formulas by converting p_{Ives} to p_{Troland} from the relation $p_T=p_I/(p_I-1)$.

Of course, it is possible to inscribe lines on the color triangle, from either set of formulas, which represent purities not physically realizable, (i.e., lines which fall outside the area bounded by the spectrum scale

and the straight line joining its extremes), since any value of r, g, and b, positive or negative, may be obtained from the formulas by substituting the right value for the purity p. Whether the purities represent physical colors or not, however, is beside the point. The formulas, either from Ives' definition or Troland's definition, give the relationship between the trichromatic representation of color data and an arbitrary "monochromatic" method of representation, and they give this relationship whether the objective stimulus invoked is real or imaginary. From the foregoing, it is evident that Ives' definition of purity is the more convenient.

In conclusion, the author wishes to acknowledge his indebtedness to Mr. I. G. Priest for his hearty coöperation.

BUREAU OF STANDARDS,
AUGUST 1925–FEBRUARY 1926.

APPENDIX

IMPORTANCE OF THE COLLINEARITY CONDITION DEMONSTRATED BY ACTUAL COMPUTATION

When any of the forms of purity formula are applied "blindly" (i.e. without regard for the condition equations beyond the accuracy dictated by experimental considerations) the result must be looked upon as so unreliable as to be almost meaningless. Priest[33] has computed from Tuckerman R, Tuckerman G, and Tuckerman B, the purities of a wide range of colors from Weaver's tabulation of "Representative Monochromatic Analyses,"[34] and finds differences in some cases as large as 0.100. In order to show that most of these differences are due to collinearity errors in the *third* place of Weaver's excitation data, the computations of Table 2 have been carried out. The wave length 440 mμ was chosen because: (1) that wave length yields the largest deviations recorded by Priest, and (2) at the same time is sufficiently far distant from the "regions of inconvenience" (Section II, Table 1) that the gross deviations found can not possibly be ascribed to simple rejection error based on the failure to meet exactly the other conditions. A reference to Table 2 and footnote will show how completely the differences are eliminated.

DERIVATION OF THE PURITY FORMULA FOR COLORS OF PURPLE HUE

Let the point (r, g, b) represent the color of the purple hue whose purity, p, is desired. Since this color is of purple hue, there exists homogeneous radiation which, when added to the non-spectral stimulus, will be a color match for the neutral stimulus. Let ρ_λ, γ_λ, β_λ be the excitation values for this homogeneous radiation. Let ρ_c, γ_c, β_c be the excitation values for one of the non-spectral stimuli complementary (in general evoking a purple not referred to by the point, r, g, b, though this point is not excluded) in hue to the spectral color. Then:

$$\rho_\lambda+\rho_c=\gamma_\lambda+\gamma_c=\beta_\lambda+\beta_c=K$$

And, from this:

$$\rho_c=K-\rho_\lambda\,; \qquad \gamma_c=K-\gamma_\lambda\,; \qquad \beta_c=K-\beta_\lambda\,.$$

[33] Priest: Loc. cit., p. 517.
[34] Troland: Loc. cit., pp. 586–587.

By adding an amount, ω, (where ω may take positive, negative or zero values) to each of ρ_λ γ_λ, β_λ it will be possible to arrive at the color represented by the point (r, g, b), hence:

$$r=\frac{\omega+K-\rho_\lambda}{3(\omega+K)-(\rho_\lambda+\gamma_\lambda+\beta_\lambda)}\ ;\quad g=\frac{\omega+K-\gamma_\lambda}{3(\omega+K)-(\rho_\lambda+\gamma_\lambda+\beta_\lambda)}\ ;\quad B=\frac{\omega+K-\beta_\lambda}{(\omega+K)-(\rho_\lambda+\gamma_\lambda+\beta_\lambda)}\ . \tag{1}$$

The coordinates of the point representing the spectral complementary of (r, g, b) are:

$$R=\frac{\rho_\lambda}{\rho_\lambda+\gamma_\lambda+\beta_\lambda}\ ;\qquad G=\frac{\gamma_\lambda}{\rho_\lambda+\gamma_\lambda+\beta_\lambda}\ ;\qquad B=\frac{\beta_\lambda}{\rho_\lambda+\gamma_\lambda+\beta_\lambda}\ . \tag{2}$$

From the experimental procedure in homo-hetero analysis with respect to colors of purple hue: (See Section IV).

$$\boldsymbol{B}_\Delta+\boldsymbol{B}_s=\boldsymbol{B}_m=\boldsymbol{B}_w\ .$$

From this point, the derivation depends upon what function of the various field brightnesses is chosen to represent the purity of the color (r, g, b). Take Ives' definition first: $p=\boldsymbol{B}_\Delta/\boldsymbol{B}_s$, $\boldsymbol{B}_\Delta$ being taken negative. The brightnesses may then be expressed in terms of the luminosity coefficients (L_r, L_g, L_b) of the elementary excitations, as follows:

$$\begin{aligned}\boldsymbol{B}_\Delta&=-(\rho_\lambda L_r+\gamma_\lambda L_g+\beta_\lambda L_b)\ .\\ \boldsymbol{B}_s&=(K+\omega-\rho_\lambda)L_r+(K+\omega-\gamma_\lambda)L_g+(K+\omega-\beta_\lambda)L_b,\\ &=(K+\omega)(L_r+L_g+L_b)-(\rho_\lambda L_r+\gamma_\lambda L_g+\beta_\lambda L_b)\ .\end{aligned} \tag{3}$$

$$\boldsymbol{B}_s=K+\omega+\boldsymbol{B}_\Delta\ . \tag{4}$$

$$\boldsymbol{B}_m=\boldsymbol{B}_w=\boldsymbol{B}_\Delta+\boldsymbol{B}_s=K+\omega+2\boldsymbol{B}_\Delta\ . \tag{5}$$

From equations (1):

$$\frac{\omega}{\rho_\lambda+\gamma_\lambda+\beta_\lambda}=\frac{r-R}{3r-1}+\frac{2\boldsymbol{B}_\Delta}{\rho_\lambda+\gamma_\lambda+\beta_\lambda}=\frac{g-G}{3g-1}+\frac{2\boldsymbol{B}_\Delta}{\rho_\lambda+\gamma_\lambda+\beta_\lambda}=\frac{b-B}{3b-1}+\frac{2\boldsymbol{B}_\Delta}{\rho_\lambda+\gamma_\lambda+\beta_\lambda} \tag{6}$$

From equation (5): $p=\boldsymbol{B}_\Delta/\boldsymbol{B}_s=\boldsymbol{B}_\Delta/(\boldsymbol{B}_w-\boldsymbol{B}_\Delta)=\boldsymbol{B}_\Delta/(K+\omega+\boldsymbol{B}_\Delta)$. To evaluate K, the constant from which the measurement of the variable, ω, may be said to start, let $p=-1$, when $\omega=0$, then: $-1=\boldsymbol{B}_\Delta/(K+\boldsymbol{B}_\Delta)$; and $K=-2\boldsymbol{B}_\Delta$.
From (5), $\boldsymbol{B}_w=\omega$, whence $p=-\boldsymbol{B}_\Delta/(\boldsymbol{B}_\Delta-\omega)$.

Transforming to the arbitrary brightness units of Section I:

$$\boldsymbol{b}_\lambda=\boldsymbol{B}_\Delta/(\rho_\lambda+\gamma_\lambda+G_\lambda)\ . \tag{7}$$

$$\boldsymbol{b}_\lambda=-(RL_r+GL_g+BL_b)\ ,\ \text{from (2) and (3)}\ , \tag{8}$$

And, from (6):

$$\frac{\omega}{\rho_\lambda+\gamma_\lambda+\beta_\lambda}=\frac{r-R}{3r-1}+2\boldsymbol{b}_\lambda=\frac{g-G}{3g-1}+2\boldsymbol{b}_\lambda=\frac{b-B}{3b-1}+2\boldsymbol{b}_\lambda\ . \tag{9}$$

Substituting: (7), (8) and (9) in the relation: $p=-\boldsymbol{B}_\Delta/(\boldsymbol{B}_\Delta-\omega)$

$$p=\frac{RL_r+GL_g+BL_b}{RL_r+GL_g+BL_b+(r-R)/(1-3r)} \qquad \text{(Judd R)}$$

$$p=\frac{RL_r+GL_g+BL_b}{RL_r+GL_g+BL_b+(g-G)/(1-3g)} \qquad \text{(Judd G)}$$

$$=\frac{RL_r+GL_g+BL_b}{RL_r+GL_g+BL_b+(b-B)/(1-3b)} \qquad \text{(Judd B)}$$

These three forms will be recognized as the Judd R, Judd G, and Judd B forms derived originally to apply to colors of spectral hue. The Ives' definition for purity, therefore, yields the same formula for all colors, spectral or non-spectral.

Taking, in turn, Troland's definition for purity of colors of purple hue: $p=B_\Delta/B_m$, the brightnesses being taken always positive, we proceed as before. Only the relations which differ from those in the previous derivation are given in the following, the corresponding equations being denoted with the subscript "a."

$$B_\Delta=\rho_\lambda L_r+\gamma_\lambda L_g+\beta_\lambda L_b\,. \tag{3a}$$

$$B_s=K+\omega-B_\Delta \tag{4a}$$

$$B_m=B_w=B_s+B_\Delta=K+\omega-B_\Delta+B_\Delta=K+\omega\,. \tag{5a}$$

$$\frac{\omega}{\rho_\lambda+\gamma_\lambda+\beta_\lambda}=\frac{r-R}{3r-1}-\frac{2B_\Delta}{\rho_\lambda+\gamma_\lambda+\beta_\lambda}=\frac{g-G}{3g-1}-\frac{2B_\Delta}{\rho_\lambda+\gamma_\lambda+\beta_\lambda}=\frac{b-B}{3b-1}-\frac{2B_\Delta}{\rho_\lambda+\gamma_\lambda+\beta_\lambda} \tag{6a}$$

From (5a):

$$p=B_\Delta/B_m=B_\Delta/(K+\omega)$$

To evaluate K, let $\omega=0$, when $p=\frac{1}{2}$[25]
Then: $\frac{1}{2}=B_\Delta/K$; and $K=2B_\Delta$
Hence from (5a): $p=B_\Delta/(2B_\Delta+\omega)$.
But from (2) and (3a):

$$b_\lambda=RL_r+GL_g+BL_b\,,\ \text{therefore} \tag{8a}$$

$$p=\frac{b_\lambda}{2b_\lambda+\omega/(\rho_\lambda+\gamma_\lambda+\beta_\lambda)}\,.$$

$$\frac{\omega}{\rho_\lambda+\gamma_\lambda+\beta_\lambda}=\frac{r-R}{3r-1}-2b_\lambda=\frac{g-G}{3g-1}-2b_\lambda=\frac{b-B}{3b-1}-2b_\lambda\,. \tag{9a}$$

Substituting:

$$p=\frac{RL_r+GL_g+BL_b}{-(r-R)/(1-3r)}$$

$$p=\frac{RL_r+GL_g+BL_b}{-(g-G)/(1-3g)}$$

$$p=\frac{RL_r+GL_g+BL_b}{-(b-B)/(1-3b)}$$

These forms are recognized to be different from the Troland forms for colors of spectral hue. The Troland definition ($p=B_\Delta/B_m$) does not, therefore, yield formal continuity.

[25] That this is the natural choice may be seen from the fact that $p=\frac{1}{2}$ in Troland's system corresponds to $p=-1$ in Ives'. We cannot set $p=1$, since $p=B_\Delta/B_m=B_\Delta/(B_\Delta+B_s)<1$, if $B_s\neq 0$.

STRAIGHT-LINE RELATIVITY IN OBLIQUE COORDINATES; ALSO ILLUSTRATED BY A MECHANICAL MODEL†

By Vladimir Karapetoff

ABSTRACT

It is shown that the fundamental space-time-velocity relations of Einstein's original restricted theory of relativity, when the observed point, line, or ray of light moves in the direction of the observers themselves, can be represented quantitatively and visualized by using two sets of oblique time-space coordinates, forming the *Lorentzian plane*. For the observer moving in the positive direction at the relative velocity q, the angle between the coordinate axes is $90°-\alpha$, for the observer moving in the negative direction the angle is $90°+\alpha$, where $\sin \alpha = q/c$; α has been named the *velocity angle*.

The concept of *intrinsic coordinates* of an event is introduced, and certain simple phenomena are expressed in these coordinates. A *spatium* is defined as a vector which can be differently resolved into space and time components by different observers. The general method is illustrated by showing graphically the Fitzgerald contraction of a length, the slowing down of a clock, the fact that two velocities always add to one less than that of light, the Doppler effect, reflection from a moving mirror, etc.

A simple *mechanical model* is described which conveys some of the foregoing relations directly to the eye and which can be used for demonstrations before a large audience.

Terms and Notation used.

a, a'	acceleration of a moving point
c	velocity of light
E_0	energy added to a system of moving particles, Eq. (66)
h	intrinsic spatium of a moving line*
i	used as a subscript means incidence; Fig. 10
l, l'	lengths
MN	in Fig. 6, a spatium, that is, a vector which may be resolved into a length and a time interval*
OB, OB_0	universal bisectors, positive and negative, Figs. 3 and 7*
OH	Fig. 7, space-time line or curve
q	relative velocity between S and S'
r	spatium vector of a point in the Lorentzian plane, Fig. 3*
r	used as a subscript means rebound or reflection
S, S'	two systems or observers moving at a relative velocity q with respect to each other; Fig. 1
T, T'	time intervals
t, t'	ordinates along the time axes
v, v'	velocities of a point
$XX'OTT'$	Lorentzian plane, Fig. 3*
x, x'	lengths plotted as abscissae
α	velocity angle, Eq. (3) and Fig. 3*

† The experimental part of the investigation upon which this article is based was supported by a grant from the Heckscher Foundation for the Advancement of Research, established by August Heckscher at Cornell University.

γ, γ' coordinate angles, between the coordinate axes and the universal bisector OB, Eqs. (4) and (5); Fig. 3*

θ orientation angle of a space-time line, Fig. 7*

λ, λ' wave lengths of light

μ, μ' values of the refraction coefficient

ν, ν' frequencies of light

ϕ orientation angle of a spatium vector, Fig. 3*

* Note 1. The quantities and definitions marked with an asterisk* are believed to be new.

Note 2. Where an unprimed quantity and a primed quantity are given side by side, as for example a and a', the notation a is understood to refer to the S system and a' to the S' system.

I. INTRODUCTION

It was Hermann Minkowski who in 1908, in his now classical paper "Raum und Zeit," first gave a graphical interpretation of Einstein's principle of relativity. Minkowsky represented time as an imaginary fourth dimension in space and showed that the fundamental relativity results could be thought of as due to a deformation of such a four-dimensional continuum by an observer's own motion. If the relative motion of two observers and the event under observation both take place along the X-axis, then for some purposes the occurrences along the Y and Z axes need not be considered. and the four-dimensional continuum is reduced to a plane relationship with coordinate axes X (length) and T (time). An observer's motion is supposed to change this set of orthogonal axes to a system of oblique axes. Later writers followed Minkowski's idea and elaborated it in some detail.[1]

The foregoing representation suffers from the fact that the time coordinate is imaginary, and also because the world of experi nce is referred to orthogonal coordinates for one observer and to oblique ones for the other. The present writer starts therefore with the assumption that the X-T two-dimensional world is to be referred to a different set of oblique coordinates for each observer, the departure from the right angle to be the same and to depend upon the relative velocity of the two observers. Moreover, time is to be considered a real coordinate and Euclidean relationships to prevail. It is shown below that with these two assumptions, (a) all the principal quantitative results of restricted relativity are obtained in a very simple manner; (b) that these results can be directly visualized; (c) that new results can be readily deduced from the simple geometric relationships which obtain with this mode of representation.

[1] See, for example, A. Brill, Das Relativitätsprinzip, Leipzig, p. 13; 1920. A. S. Eddington, Space, Time and Gravitation, Cambridge Univ. Press, p. 45; 1921.

II. STATEMENT OF THE PROBLEM

The fundamental relativity problem in its simplest form is stated in Fig. 1. Two systems, S and S', move at a uniform velocity q relatively to each other, along the direction XX'. An observer on S considers himself at rest, and for him the system S', with an observer on it, is moving at a velocity q to the right. The observer on S' considers himself at rest and for him the S system, with its observer, is moving at the same velocity q to the left. Both observers agree that the relative velocity of the two systems is q. There is nothing whatever in the whole arrangement to indicate which system is at rest. In matter of fact, the very idea of absolute rest or of an absolute reference origin is adverse to the principle of relativity.

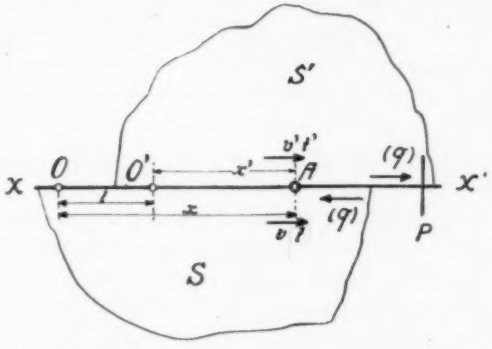

FIG. 1. *Relative motion of systems S and S' along XX'.*

A material point A, independent of either system, is in motion along XX', and both observers record this motion in the manner shown in Fig. 2. Each observer has his own origin (marked O and O' in Fig. 1), and they use standards of length and of time which have been checked to agree when S and S' are at rest with respect to each other. The S observer records the motion of point A in the form of the curve MN, giving the positions of A at various instants of time with respect to his own origin O. The S' observer does the same with respect to his origin and his clock; in fact, throughout this article the two observers are treated alike and there is no "stationary" or "moving" observer. Symbols with and without the "prime" mark (for example S and S') have been used by many writers on relativity and this notation is preserved here, without implying in the least that the results of the S' observer are in any way "warped" and those of S are "true" results.

Fig. 2 merely shows the general character of the record and refers to either observer.

As a result of the independent measurements of the same event by the two observers, we get two different curves MN. The problem is to find the relationship between these two curves. In other words, knowing the space-time curve observed by S, and knowing the velocity

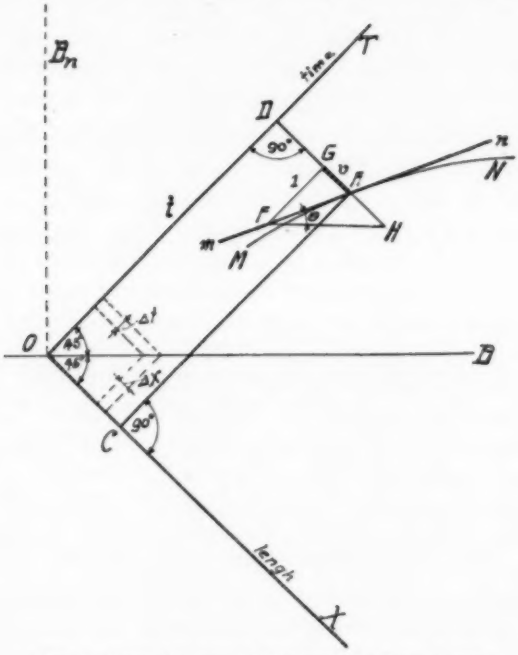

FIG. 2. *Space-time curve MN referred to orthogonal axes.*

q of S' with respect to S, it is required to determine theoretically the curve that would be measured by S'. To put both observers on the same footing, we might say that it is required to find the general relationship between the time, space, and velocity measurements of two observers whose relative velocity is q.

III. AN EVENT IN ORTHOGONAL COORDINATES

In order to simplify the graphical construction and the formulae, we shall make the following two assumptions:

(1) There is a clock at O and one at O' (Fig. 1). Both clocks are set to indicate zero at the instant when O and O' coincide. This simplifies the reckoning of time in the two systems.

(2) The units of time and space are so chosen that a ray of light in vacuum covers a unit distance in unit time. In other words, the velocity of light in empty space is taken as unity. For example, if the unit of time is one second, then the unit of length must be equal to 300,000 km. If the unit of time is one n-th of a second, the unit of length must be one n-th of 300,000 km. Results obtained in this system of units can be readily recomputed for any other desired system.

In this system of units, the bisector OB (Fig. 2) represents the motion of a point at the velocity of light, the point passing through the origin at the time zero. This is because for such a motion an increment Δt of time is numerically equal to the corresponding increment of space Δx. The velocity of the point, or the ratio Δx to Δt is equal to 1. Any line parallel to OB also represents a motion at the velocity of light, only the corresponding point does not pass through the origin at the time zero.

The velocity v of a point represented by the curve MN is variable, and at any instant, such as A, is numerically equal to the trigonometric tangent of the angle which the tangent mn to the space-time curve MN forms with the T-axis. This is because $v=\Delta x/\Delta t$. FGH is an isosceles right-angle triangle, with $FG=GH=1$. Its sides are parallel to the coordinate axes and to the bisector OB respectively. The segment GA is numerically equal to the velocity v at A, expressed as a fraction of the velocity of light. At N, where the tangent to the path is practically parallel to OB, the velocity of the point approaches that of light.

One of the postulates of the theory of relativity is that no velocity greater than that of light can be attained by a material particle or by an electromagnetic wave. Hence, for a real material point moving in the direction from X to X' (Fig. 1) the tangent to the space-time curve MN can only have inclinations intermediate between those parallel to OT ($v=0$) and parallel to OB ($v=1$). For a motion in the negative direction the slope is limited between the parallel to OT ($v=0$) and that parallel to the "negative" bisector OB_n ($v=-1$).

IV. AN EVENT IN OBLIQUE COORDINATES; THE LORENTZIAN PLANE

Since time intervals (and not distances) are plotted along the OT-axis (Fig. 2), there is no particular reason for which this axis should be perpendicular to OX. In fact, each observer could select an arbitrary angle between OX and OT and plot his observations of the motion of point A (Fig. 1) using a set of oblique coordinates. Two sets of such oblique coordinates are shown in Fig. 3, with a common bisector OB.

The dotted axes X_0OT_0 represent a rectangular set, the same as in Fig. 2, and are drawn only in order to show the relative position and the angles of the two actual oblique sets used.

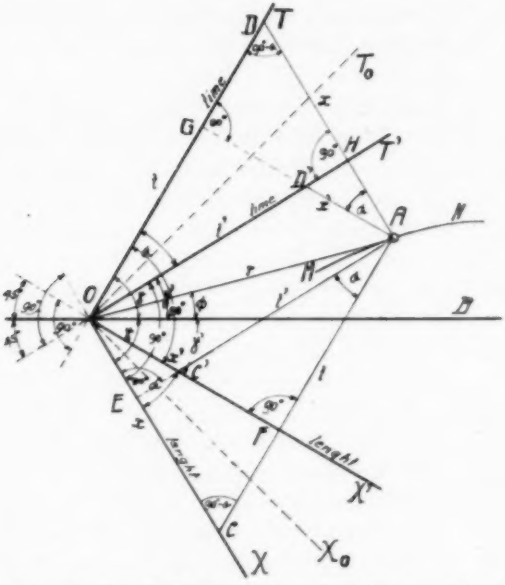

FIG. 3. *Space-time curve MN referred to two sets of oblique axes forming a Lorentzian plane.*

The findings of the observer S will be recorded in the coordinate system XOT, with the central angle

$$XOT = 90° + \alpha \tag{1}$$

while for the S' observer the axes OX' and OT' will be used with the angle

$$X'OT' = 90° - \alpha \tag{2}$$

between them. Furthermore, the angle α is such that

$$\text{Sin } \alpha = q \tag{3}$$

where q is the relative velocity of S and S', expressed as a fraction of the velocity of light. The angle α will be referred to as the *velocity angle*. The angles marked 45° in Fig. 2, are denoted in Fig. 3 by γ and γ' respectively, where

$$\gamma = 45° + 0.5\alpha \tag{4}$$

$$\gamma' = 45° - 0.5\alpha \tag{5}$$

We shall call γ and γ' the *coordinate angles*, and the bisector OB, from which they are measured, the *universal bisector*, since the position of OB remains the same when the velocity angle α is varied. Eqs. (4) and (5) may also be written in the form

$$\gamma+\gamma'=90^\circ \tag{6}$$

$$\gamma-\gamma'=\alpha \tag{7}$$

We shall assume that the world in which the observers S and S′ are located has such time-space properties that a curve, such as MN (Fig. 3), which correctly represents an event for the S observer, in the oblique coordinate system XOT, also correctly represents it for the S′ observer in the oblique system X′OT′, provided that Eqs. (1), (2), and (3) are satisfied. This means that when S measures say an instantaneous flash of light, A (Figs. 1 and 3), occurring at a distance $OC=x$ from his origin, and at the instant $OD=t$, the S' observer records *the same flash* as having taken place at the distance $OC'=x'$ from his origin and at the instant of time $OD'=t'$. The same relationship between the coordinates applies to any other point on the event curve MN. Of course, x and x' are in reality measured along the same line XX' (Fig. 1), and the construction in Fig. 3 is merely intended to bring out the scalar numerical relationships of the quantities involved, without any question as to their direction in space. By assumption, all events under consideration take place along the same line XX' along which the relative motion of the two systems takes place. The arrangement of the axes of coordinates shown in Fig. 3 is fundamental in this article, and will be referred to as the *Lorentzian Plane*.

There is nothing illogical *per se* in the assumptions represented by Fig. 3, and we can proceed to deduce analytically the relationship between the abscissae x and x' of a point, between the corresponding ordinates t and t', between the velocities v and v' as measured by the two observers, etc. It will be shown that the results so obtained are the same as those deduced analytically in Einstein's theory, or by means of Lorentz's transformation of coordinates. Hence, we may say that Fig. 3 is a correct graphical interpretation of the restricted theory of relativity. It is conceded that the theory of relativity gives correct results for certain physical facts, such as the Doppler effect, reflection from a moving mirror, Fizeau's experiment, etc., and to that extent it may be said to represent the actual state of things. Hence, to the same degree of certainty, Fig. 3 represents uni-dimensional time-space relations *in our world*. In our everyday experience the velocity q,

expressed as a fraction of the velocity of light, is exceedingly small. Therefore, according to Eq. (3), the velocity angle α is also very small, and the two systems of axes practically coincide with each other and with the orthogonal system X_0OT_0. For this reason, there is nothing in our every day experience, or even in a great majority of scientific measurements, to indicate the peculiar time-space relationships shown in oblique coordinates. These relationships become necessary only in the interpretation of certain high-velocity physical phenomena mentioned above.[2]

V. THE CORRESPONDING COORDINATES OF A POINT IN THE TWO SYSTEMS; THE INTRINSIC COORDINATES

Our first problem shall be to find an analytical relationship between x and x' for the same event A (Fig. 3). We extend the ordinate $AC'=t'$ to its intersection with OX at E, and find

$$OE=OC-EC$$

or
$$x' \operatorname{Cos} \alpha = x - t \sin \alpha \tag{8}$$

We also have

$$OF=OC'+C'F$$

or
$$x \cos \alpha = x' + t' \sin \alpha \tag{9}$$

In Eqs. (8) and (9) we now substitute for $\sin \alpha$ its value from Eq. (3), and for $\cos \alpha$ the corresponding value

$$\cos \alpha = \sqrt{1-q^2} \tag{10}$$

The foregoing equations then become

$$x' = (x-tq)/\sqrt{1-q^2} \tag{11}$$

$$x = (x'+t'q)/\sqrt{1-q^2} \tag{12}$$

These expressions agree with the well-known relationships found in any elementary mathematical book on relativity. By a similar reasoning we find from the triangles OHD and DAG

$$t' = (t-xq)/\sqrt{1-q^2} \tag{13}$$

$$t = (t'+x'q)/\sqrt{1-q^2} \tag{14}$$

These relationships are also among those upon which the accepted theory of relativity is based. Thus, for any assumed values of distance and time for one of the observers, the construction in Fig. 3 gives such values of these quantities for the other observer, as would be obtained by computation in accordance with the usual relativity formulae.

[2] A similar principle of oblique coordinates, for representing a phenomenon as it appears to two observers in relative motion, has been applied by the present writer to plane electromagnetic waves; See the Physical Review, *23*, p. 239; 1924.

An instantaneous event, such as A, may also be represented by its *intrinsic coordinates*, $r=OA$, and angle $\phi=AOB$. Since OB remains the same for all coordinate systems (that is, with various values of q), the polar coordinates r and ϕ define A uniquely, although not directly in terms of space and tine. The vector r is a so-called *spatium*, defined and discussed below. The coordinates of A in either system may be expressed through its intrinsic coordinates. Namely, from the triangle ODA we have

$$x/r=\sin(\gamma-\phi)/\cos\alpha$$

so that

$$x=r\sin(\gamma-\phi)/\cos\alpha \tag{15}$$

Similarly, from the triangle OCA

$$t=r\sin(\gamma+\phi)/\cos\alpha \tag{16}$$

For the S' system, analogous expressions hold true, with γ' substituted in place of γ. Thus

$$x'=r\sin(\gamma'-\phi)/\cos\alpha \tag{17}$$

$$t'=r\sin(\gamma'+\phi)/\cos\alpha \tag{18}$$

By means of the foregoing expressions, the coordinates x and t of an instantaneous event in either system can be written directly from the intrinsic coordinates r and ϕ of the same event, without using the coordinates of the other system. In applications, it is convenient to remember that $\gamma+\gamma'=90°$, so that $\sin(\gamma'+\phi)=\cos(\gamma-\phi)$, etc. Substituting the values of x, t, x', and t' from Eqs. (15), (16), (17), and (18) in Eqs. (8) and (9), identities are obtained. This means that Eqs. (8) and (9) may be looked upon as results of elimination of the intrinsic coordinates r and ϕ from Eqs. (15) to (18).

Let x and t in Eqs. (15) and (16) be given and let the velocity q be known so that the angles γ and α are also known. Let it be required to solve these equations for the intrinsic coordinates r and ϕ. To eliminate r, we divide one equation by the other, and obtain

$$t/x=\sin(\gamma+\phi)/\sin(\gamma-\phi).$$

Using the lemma in the Appendix, this equation is readily transformed into

$$\tan\phi=\tan\gamma(t-x)/(t+x) \tag{19}$$

The spatium r can be directly determined from the triangle OAC; namely

$$r^2=t^2+x^2-2xt\sin\alpha \tag{20}$$

Formulae (19) and (20) also hold true for the "primed" letters, that is, for the system S'.

An alternative form of Lorentzian plane construction is shown in Fig. 3a. The coordinate axes OX and OT of the S observer are drawn at an angle γ about the positive bisector OB, as in Fig. 3, while those of S' are drawn at an angle γ' with respect to the negative bisector OB_n. Since $\gamma+\gamma'=90°$, the axis OT' coincides with OT and OX' coincides with the negative extension of OX. With this arrangement of the axes,

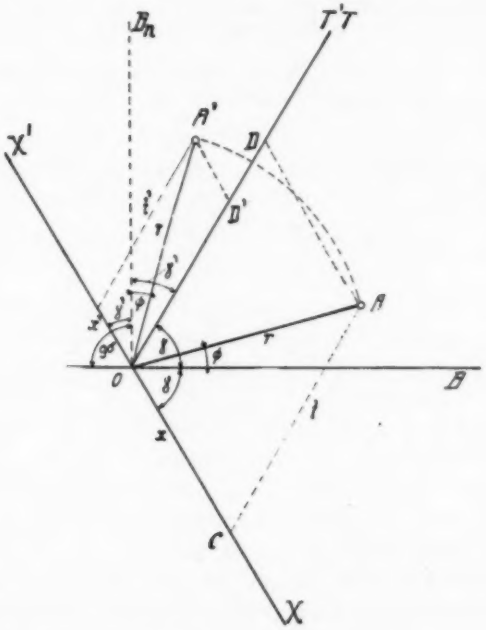

FIG. 3a. *A variant of Fig. 3.*

it is necessary to show each event twice. For example, the event corresponding to the point A is represented once by its intrinsic coordinates, the radius vector r and the angle ϕ which r forms with OB; and then it is also necessary to show the same event at A', at the same distance r from the origin, but at the angle ϕ with the negative bisector OB_n.

The x and x' coordinates of A are in this case parallel to each other, and so are the coordinates t and t'. This has a certain advantage for a visual comparison of their lengths; moreover, various lines obscure

each other less than in Fig. 3. On the other hand, this form of Lorentzian plane has a disadvantage in that each event has to be shown twice. In this article only the arrangement of the axes shown in Fig. 3 is used.

VI. SPATIUM, APPARENT REDUCTION IN LENGTH, AND SLOWING DOWN OF A CLOCK

The vector OA in Fig. 3 is neither a length nor an interval of time, but a combination of the two. To the S observer it represents an instantaneous event (say a flash of light) characterized by the distance x from his origin and by the interval of time t. To the S' observer the vector OA stands for a different distance x' and a different interval of time t'. For a different relative velocity q, that is, for a different angle α, the resolution of OA into space and time is also different. We shall call a vector such as OA, that is, any definite segment of a line in the Lorentzian plane, that may be part time and part space, a *spatium*. Roman writers used this Latin word to designate either a length in space or an interval of time.

There are two statements usually made in works on relativity that puzzle the beginner, namely (a) clocks in the "other" system seem to run "slow"; (b) lengths in the "other" system seem shorter (Fitzgerald contraction). This means that for an S observer a clock stationary with respect to the S' system seems to run slow, while for the S' observer a clock ridigly attached to the S system also seems to run slow, even though the clocks have been properly adjusted to run at the same rate before the relative motion of the two systems began. Similarly, a meter stick attached to the S' system in the direction of its motion (that is, along XX' in Fig. 1) will appear shorter than one meter when measured by the S observer. On the other hand, a meter stick similarly placed in the S system will be found to be shorter than one meter when measured by the S' observer. The ratio of both the lengths and the time intervals is equal to $\sqrt{1-q^2}$.

These results become almost self-evident when represented in the Lorentzian plane (Fig. 4). Let a clock be placed at the origin O' (Fig. 1) of the S' system, and let another clock be placed in the S system at a distance l from the origin O. When the two clocks are exactly opposite each other, both are read, and a discrepancy is found, even though both clocks indicated zero when O was in coincidence with O'. The event under consideration is represented in Fig. 4 by the spatium $OE=t'$. This is the time read by the S' observer. For the S

observer the same spatium must be resolved parallel to his axes into a time interval or component $OF=t$ and a length component $FE=l$. It will be readily seen that $t>t'$, and that

$$t'=t\ \cos\ \alpha=t\ \sqrt{1-q^2} \tag{21}$$

Moreover

$$l=t\ \sin\ \alpha=tq \tag{22}$$

The latter result was to be expected, because l is the distance traversed by O' within time t, moving at the velocity q. The interesting part of

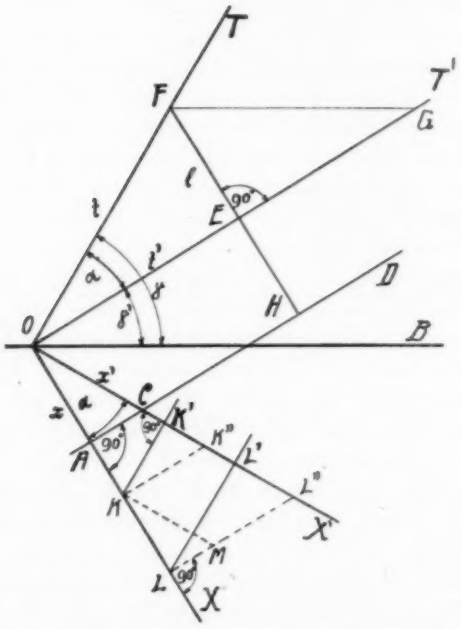

FIG. 4. *Fitzgerald contraction and apparent slowing down of a clock.*

the peculiar relationship expressed by Eq. (21) is that when S' reads the instant $OF=t$ on the clock of the S observer, he reads OG on his own clock and finds the distance OO' between the origins to be equal to FG. Thus, he also comes to the conclusion that the clock in the "other" system is "slow."

Consider now an event represented by two parallel lines, OT' and AD (Fig. 4). To the S' observer the straight line OT' simply stands for

his own origin O', because $x'=0$ for any value of t'. The line AD for him stands for a stationary point at the distance $OC=x'$ from the origin. For the S observer, OT' represents a point moving at the velocity q; AD is for him another point moving at the same velocity. To find the distance between these two points at any instant t, as measured by the S observer, we draw the line FH parallel to OX, and find $EH=OA=x$.

Thus, the observer S' finds the distance $OC=x'$ between his two stationary points, while the S observer actually measures a smaller distance $OA=x$ *between the same points*, simply because the S' system is in motion with respect to him. From the triangle OAC

$$x=x' \cos \alpha = x' \sqrt{1-q^2} \tag{23}$$

which agrees with the so-called Fitzgerald contraction coefficient of a moving body. A fixed length in the S system, such as KL, if measured by the S' observer, will be reported as a smaller length $K'L'=KL \cos \alpha$.

If, however, KL represents an *instantaneous* event, for example two simultaneous flashes of light in the S system, the interpretation of this event by the S' observer will be entirely different from that of a *continuous* event considered above. The S' observer will resolve the spatium KL into its time and length components, as shown by the dotted lines, and will report that the two flashes took place at the distance $K''L''$ from each other and that the flash at L occurred a time interval LM later than the K flash. In this case the length measured by the observer in the "other" system is greater, instead of being smaller than that measured by the "own" observer.

VII. SOME APPLICATIONS

The following two specific cases become very simple when represented in the Lorentzian plane, and their discussion may help to understand the advantages and the use of this plane.

Let the relative velocity between S and S' be almost equal to that of light (Fig. 5) so that α is almost equal to 90°. The XOT coordinates are then expanded almost into a straight line, while the $T'OX'$ coordinates are contracted so as to form small angles with the bisector OB. Consider the spatium OC. To the S' observer it represents a comparatively small interval of time. To the S observer it represents a long interval of time $OA=t$ and a long distance $CA=x$. This is sometimes expressed in a popular inaccurate way by saying that a person could remain eternally young by traveling at the velocity of light. By travelling at a velocity greater than that of light he would seem to the other observer to grow younger and younger.

The so-called "cigar example" is shown in Fig. 6.[3] A person in the S system measures the useful length l of a cigar, lights it at a known instant of time, places it parallel to XX', smokes it at a uniform rate, and notes the final instant of time. In the Lorentzian plane this event can be represented by the spatium MN. Resolving MN into its components parallel to the axes OX and OT, we find that the initial length

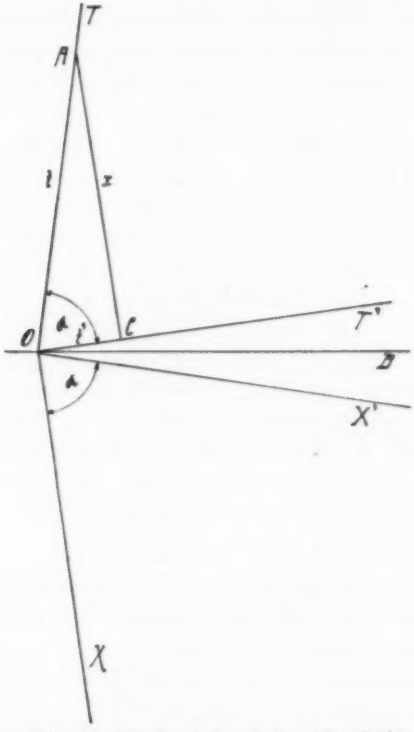

FIG. 5. *Time distortion when the relative velocity of S and S' is nearly equal to the velocity of light.*

of the cigar was l, and the time consumed in smoking it was t. The S' observer also measures the length of the same cigar and the time consumed by S in smoking it, but finds different values, l' and t'. These values are obtained by resolving MN into components parallel to the axes OX' and OT'.

[3] Eddington, Space, Time and Gravitation, p. 24; 1921.

If we assume that our world is constituted in accordance with the transformation of time-space coordinates as illustrated in the preceding sketches, then we have to accept the consequences, however unusual they may seem in the light of our everyday experience. It must be remembered that for ordinary low velocities with which we deal in daily life, the Lorentzian transformation gives "common sense"

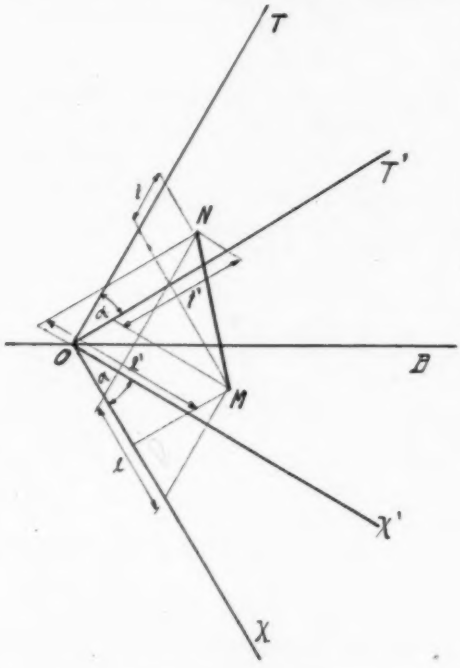

FIG. 6. *A point in uniform motion between two given points in space (the cigar problem).*

results. In any event, the present article is in no way a defense or a new proof of Einstein's theory, but only an attempt at a graphical illustration of its fundamentals.

VIII. VELOCITY MEASUREMENT

Let Fig. 7 represent a Lorentzian plane and let OH be a space-time line which represents a uniform motion of some material point A along XX' (Fig. 1). For the sake of simplicity, let us assume the point A to be at the common origin of S and S' at the time zero. The time-space

curve of any other point moving at the same velocity along XX' can be represented in Fig. 7 by a line parallel to OH, for example by MN.

By definition of uniform velocity, it is equal to the path covered in unit time. Hence, if we lay off on OT a length OF equal to unit time and draw FD parallel to OX, the length FC will represent the velocity v

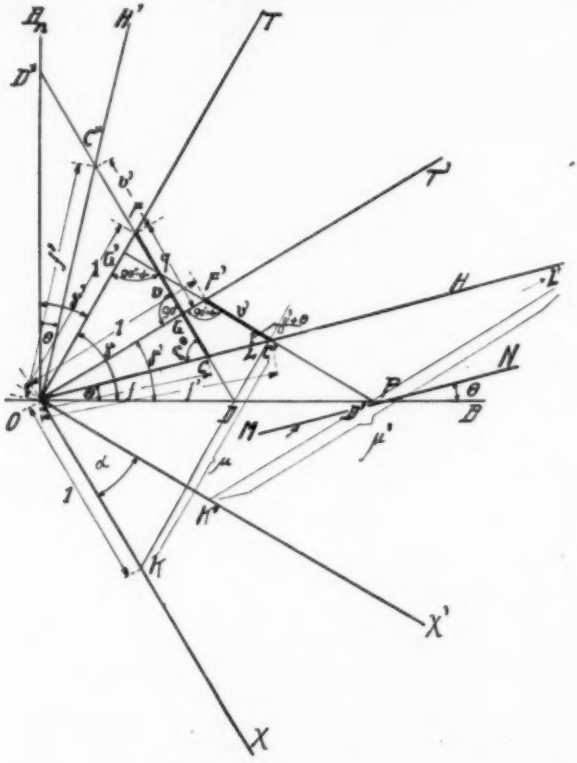

FIG. 7. *Velocity measurement.*

of the point A, measured by the S observer. Similarly, if we lay off on OT' a length OF' equal to unit time and draw $F'D'$ parallel to OX', the length $F'C' = v'$ will represent the velocity of the same point A, measured by the S' observer. It is understood, of course, that both v and v' are in reality directed along XX' (Fig. 1), and that Fig. 7 merely gives the numerical values of these velocities.

The direction OH can be defined by the angle θ which it forms with the universal bisector OB. This angle will be referred to as the *orientation angle* and is an intrinsic coordinate of the line OH. The space-time line MN is determined by the same angle θ (because it corresponds to the same velocity) and by the spatium OP along the bisector.

It is convenient to express both v and v' through the orientation angle θ. From the triangle OFC we have:

$$v = \sin(\gamma - \theta)/\sin(\gamma + \theta) \tag{24}$$

Analogously, from the triangle $OF'C'$,

$$v' = \sin(\gamma' - \theta)/\sin(\gamma' + \theta) \tag{25}$$

By eliminating θ from these two equations, the usual relationship between v and v' is obtained, such as is given in mathematical works on relativity. In order to obtain this relationship, we use the lemma given in the Appendix, and rewrite Eqs. (24) and (25) in the form

$$(1+v)/(1-v) = \tan\gamma/\tan\theta \tag{26}$$

$$(1-v')/(1+v') = \tan\theta/\tan\gamma' \tag{27}$$

Multiplying these equations term by term, to eliminate $\tan\theta$, we get

$$(1+v)(1-v')/[(1-v)(1+v')] = \tan^2\gamma.$$

But, from Eq. (4), and using a familiar trigonometric transformation,

$$\tan\gamma = \tan(45° + 0.5\alpha) = [(1+\sin\alpha)/(1-\sin\alpha)]^{0.5}$$

Hence,

$$(1+v)(1-v')/[(1-v)(1+v')] = (1+q)/(1-q) \tag{28}$$

After actually performing the multiplications, and solving for v, this gives

$$v = (v'+q)/(1+qv') \tag{29}$$

which is the form in which the relationship between v and v' is usually given in works on the theory of relativity.

Taking natural logarithms of both sides of Eq. (28), dividing by 2, and replacing $0.5 \log[(1+v)/(1-v)]$ with $\tanh^{-1} v$, with corresponding substitutions for v' and q, we get

$$\tanh^{-1}v - \tanh^{-1}v' = \tanh^{-1}q \tag{28a}$$

This relationship has been pointed out by A. A. Robb who also proposed the name *rapidity* for the expression $\tanh^{-1}v$. It will be seen that rapidities are added directly, while the corresponding velocities are combined in accordance with Eq. (29).

For some purposes it is more convenient to express v through θ in a somewhat different manner. Since the origin O' (Fig. 1) moves at a velocity q with respect to O, the direction OT' in Fig. 7 represents the

motion of O' as it appears to the observer on S. Hence $FG=q$. From the right-angle triangle OCG we have

$$(v-q)/\cos\alpha=\tan(\gamma'-\theta)=\cot(\gamma+\theta) \tag{30}$$

By analogy, from the triangle $OC'G'$,

$$(v'+q)/\cos\alpha=\tan(\gamma-\theta)=\cot(\gamma'+\theta) \tag{31}$$

When the direction OH lies between OB and OT', both v and v' are positive and less than the velocity of light. When OH lies between OT' and OT, v is positive and v' is negative. When OH lies between OT and the negative bisector OB_n, both v and v' are negative. For a point moving at the velocity of light in the positive direction, the time-space line coincides with OB; for one moving in the negative direction it coincides with OB_n. Since one of the postulates of the theory of relativity is that no material particle or electromagnetic wave can move at a velocity greater than that of light, all the possible values of the orientation angle θ are comprised between 0 and 90°.

Let a point be moving at the velocity of light with respect to the system S', in the positive direction. Then its space-time line is OB, and S' observer measures the velocity $F'D'=1$. The S observer measures the velocity FD, also equal to one, even though the velocity of the point with respect to himself, according to our ordinary notions, is $1+q$ and not 1. This simply illustrates the fact that in the theory of relativity velocities are not "simply added," but must be combined in accordance with Eq. (29) or its equivalent construction in Fig. 7. This construction also explains the following statements made in works on relativity; (a) any velocity less than that of light, combined with that of light, gives a velocity still equal to the velocity of light; (b) any two velocities, each of which is less than that of light, give a combined velocity less than that of light; (c) the velocity of light in vacuo is the same for all observers. The second statement is directly evident from an inspection of Fig. 7. No matter what the values of v' and of $q=\sin\alpha$ are (provided that each is less than 1) v is always less than 1. The third statement is intrinsically contained in the arrangement of the oblique axes named the Lorentzian plane.

Fig. 7 also illustrates the famous experiment by Fizeau in which a beam of light was made to pass through a rapid stream of water and was viewed by a stationary observer. The velocity of light in water is much lower than 1; let v' represent this velocity relatively to the moving water, and let q be the velocity of the water. Then v is the velocity of the same beam of light with respect to the stationary ob-

server. It is of interest to note that if we draw through D a line parallel to OT, then its segment KL is numerically equal to the refractive index, from the point of view of the stationary observer. Namely, the traingles OFC and OKL are similar, and we have

$$OF/FC = KL/OK$$

or

$$1/v = KL.$$

But by definition, the refractive index μ of a medium is equal to the ratio of the velocity of light in vacuo to that in the medium. Hence

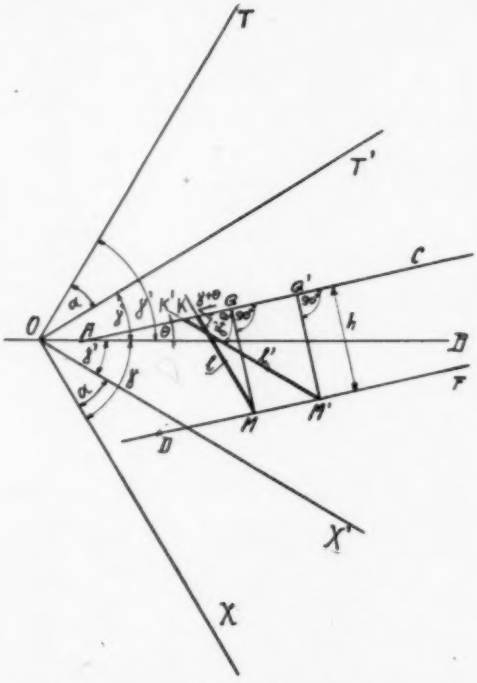

FIG. 8. *Relative lengths and the intrinsic spatium of a moving line.*

$KL = \mu$. By repeating a similar construction with respect to the axes $X'OT'$, we find the value $K'L' = \mu'$ of the refractive index that would be found by an observer stationary with respect to the water.

Should it be desired to show the velocity v' in the S system, it is only necessary to extend the line DF upward and to draw OH' at the angle θ to the negative bisector OB_n. It will be found that the triangles $OF'D'$ and OFD'' are congruent and that $FC'' = F'C' = v'$. An ad-

vantage of this mode of representation of v' is that v and v' appear on the same straight line and are easier to be compared by the judgment of the eye. See the alternative form of Lorentzian plane, at the end of Art. 5 above.

IX. A MOVING LINE AND THE DOPPLER EFFECT

Let AC (Fig. 8) be the space-time line of a point moving at a uniform velocity determined by the orientation angle θ. Let DF be a line parallel to AC, thus representing the motion of another point at the same velocity as the first. These two points, moving at the same velocity, define the extremities of a moving line of constant length. Let this line and its movement be independent of either S or S' system. The velocity of motion of such a line, as it appears to the two observers, is treated under 8 above. It remains to determine the apparent length of this line as measured by the two observers.

The line KM, drawn parallel to OX, gives the length of the line, l, such as would be measured by the S observer. The S' observer will find the length of the same line equal to $K'M'=l'$, where $K'M'$ is parallel to OX'. Instead of determining directly the ratio of l to l', we shall first express each of these through the so-called *intrinsic spatium of the line*, h. This is the minimum perpendicular distance between AC and DF. At no relative velocity q whatever can an observer find either the length of the moving line or the corresponding time interval to be less than h, and for this reason it is proposed to call h the intrinsic spatium of the line. From the triangle KMQ we have

$$l=h/\sin(\gamma+\theta) \tag{32}$$

From the triangle $K'M'Q'$, by analogy,

$$l'=h/\sin(\gamma'+\theta) \tag{33}$$

Thus
$$l/l'=\sin\ (\gamma'+\theta)/\sin(\gamma+\theta) \tag{34}$$

It is also of interest to determine the interval of time T which it takes the whole moving line to pass by the S observer (Fig. 9). This is done by drawing DH parallel to OT. From the triangle DHI

$$T=h/\sin(\gamma-\theta) \tag{35}$$

By analogy, for the S' observer,

$$T'=h/\sin\ (\gamma'-\theta) \tag{36}$$

so that

$$T/T'=\sin\ (\gamma'-\theta)/\sin\ (\gamma-\theta) \tag{37}$$

The length of the line, l, divided by the velocity v at which it is moving, should give the interval of time T during which the line completely moves past the observer. Hence, the value of l according to Eq. (32), divided by the value of v given by Eq. (24), should give the expression (35) for T. The fact that this is so constitutes a check on the formulae and on the geometric construction.

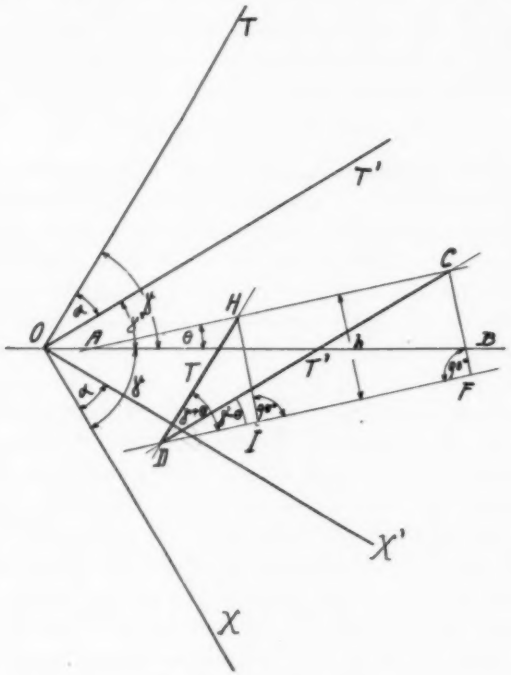

FIG. 9. *Relative values of time intervals which it takes for a line of a given spatium and velocity angle to pass by the two observers.*

As a particular case of the above formulae, let the moving line be a wave length λ of some monochromatic light. In this case both AC and DF are parallel to the bisector OB, and $\theta = 0$. Eq. (34) is simplified to

$$\lambda/\lambda' = \sin\gamma'/\sin\gamma = \tan\gamma' \tag{38}$$

Since the velocity of light is the same for both observers, the frequencies ν and ν' of the wave, measured by the two observers, will be inversely as the wave lengths. We thus have

$$\nu'/\nu = \tan\gamma' \tag{39}$$

But $\tan \gamma' = \tan (45° - 0.5\alpha) = (1 - \sin \alpha)/\cos \alpha$, so that

$$\nu'/\nu = (1-q)/\sqrt{1-q^2} \tag{40}$$

The phenomenon of the apparent change in the wave length of light, due to the observer's motion, is known as the Doppler effect, and Eq. (40) gives its value. This expression checks with those found in books on relativity and thus shows the correctness of the construction used in Fig. 8. In fact, this figure is a visual representation of the Doppler effect in the general case when the velocity of the wave is less than that of light in empty space.

X. REFLECTION FROM A MOVING OBJECT

As a further application of the foregoing principles, consider the following phenomenon: A rigid plane P (Fig. 1) is fastened to the S' system and consequently moves along the XX' axis at a uniform velocity q with respect to the S system. A perfectly elastic small sphere moves along the XX' axis at the velocity v_i (incidence) with respect to the S system, rebounds from P, and then moves back at a velocity v_r (rebound or reflection) relatively to the S system. It is required to find a relationship between v_i and v_r.

This phenomenon is represented in the Lorentzian plane in Fig. 10. The plane P being rigidly connected with the S' system, the angle α is determined by Eq. (3). Let AO be the time-space line of the motion of the sphere before the collision with P, and let OC be that after the collision. The velocities are shown as in Fig. 7. Since P is stationary with respect to S', the S' observer will find the velocity of the sphere, v', before the collision equal and opposite to that after the collision. Therefore, OA and OC must be so drawn that DE and EF be of equal length and situated on the different sides of the zero axis OT'.

For the S observer the velocities v_i and v_r are different from each other, and may be found by drawing CH parallel to OX. The diagram then gives a visual representation of the whole phenomenon. Knowing one of the velocities, the others can be determined graphically or analytically. For the analytical solution Eq. (27) can be used; namely, for the incident velocity in the S' system we have

$$(1-v')/(1+v') = \tan\theta_i/\tan\gamma'.$$

For the velocity $-v'$ of rebound in the S' system we have

$$(1+v')/(1-v') = \tan\theta_r/\tan\gamma'.$$

Multiplying these equations term by term to eliminate v' from the result, we get

$$\tan\theta_i \tan\theta_r = \tan^2\gamma' = \cot^2\gamma \tag{41}$$

If the velocity v_i is given, then applying Eq. (24) we get

$$v_i = \sin(\gamma-\theta_i)/\sin(\gamma+\theta_i)$$

from which, using the lemma in the Appendix, we have

$$\tan\theta_i = \tan\gamma(1-v_i)/(1+v_i) \tag{42}$$

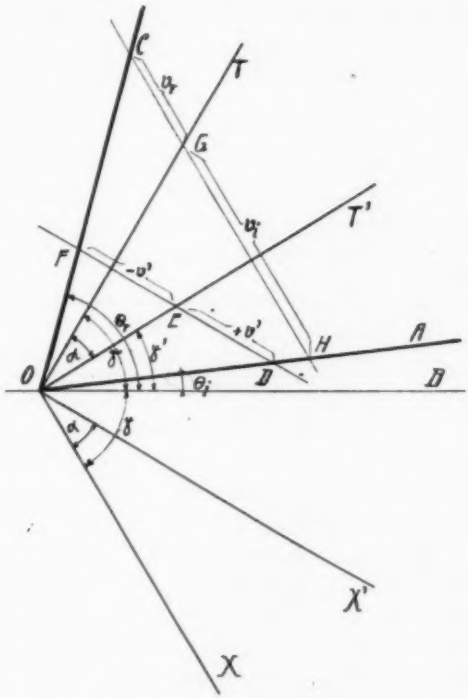

FIG. 10. *Elastic rebound from a moving plane.*

Knowing $\tan\theta_i$, the value of θ_r can be found from Eq. (41), and then v_r computed from the formula

$$v_r = \sin(\gamma-\theta_r)/\sin(\gamma+\theta_r).$$

Combining Eqs. (41) and (42), we can also write

$$(1-v_i)(1-v_r)/[(1+v_i)(1+v_r)] = (1-q)/(1+q) \tag{43}$$

As another similar problem, consider P (Fig. 1) to be a plane mirror moving with respect to S at a velocity q, in the direction from X to X'. A beam of monochromatic light is incident upon this mirror in the direction from X' to X. The mirror being stationary with respect to S', the S' observer will find the wave lengths λ' of the incident and reflected light equal to each other. The S observer, on the other hand, will find the wave length λ_r of the reflected light smaller than the wave length λ_i of the incident light, due to the motion of the mirror. It is

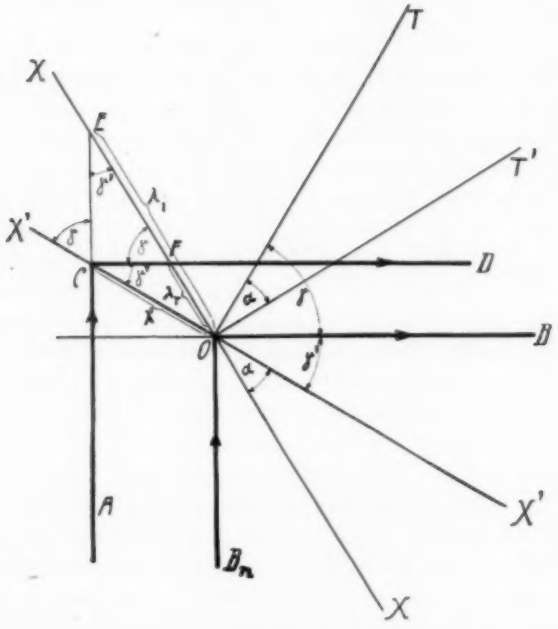

FIG. 11. *Reflection of light from a moving mirror.*

required to find the ratio of λ_r to λ_i, or its reciprocal, the ratio of the corresponding frequencies, ν_r/ν_i.

In the Lorentzian plane (Fig. 11) a point in the incident wave, moving at the velocity of light in the negative direction, is represented by the negative bisector B_nO. This point passes through the coincident origins O and O' of the two systems at the instant $t=t'=0$. Another point of the same wave, distant by one wave length, can be represented

by a line AC parallel to B_nO. After the reflection, the two points move in the positive direction, and therefore their time-space curves are represented by the positive bisector OB and by the line CD parallel to it. It will be seen from the sketch that the wave length $OC=\lambda'$, or the distance between the two moving points, is the same for the incident and for the reflected beam, when measured parallel to the X' axis. On the other hand, the S observer who measures the wave lengths parallel to the X axis, will find the wave length of the incident ray, $\lambda_i=OE$, and that of the reflected light, $\lambda_r=OF$. From the triangle OCE we have

$$\lambda'/\lambda_i=\sin\gamma'/\sin\gamma \tag{44}$$

From the triangle OCF

$$\lambda_r/\lambda'=\sin\gamma'/\sin\gamma \tag{45}$$

Multiplying these equations term by term, gives

$$\lambda_r/\lambda_i=\tan^2\gamma' \tag{46}$$

The frequencies ν_i and ν_r are inversely as the wave lengths. Consequently, using the same trigonometric transformation as in the deduction of Eq. (28), we get

$$\nu_r/\nu_i=\tan^2\gamma=(1+q)/(1-q) \tag{47}$$

The latter expression agrees with the one given for the same phenomenon in works on mathematical physics.[4]

XII. MECHANICAL MODEL

A convenient kinematic device for a study of the foregoing relationships, and which permits to demonstrate them to a large audience, is shown in Figs. 12 and 13[5]. Its principal parts correspond to the lines shown in Figs. 3, 4, and 7, and are denoted accordingly in Fig. 12. The bars may be of wood, steel, celluloid, or of any other suitable material. Since the axes OT and OX' are perpendicular to each other at any value of angle α, the corresponding bars are permanently attached to each other at right angles. The same is true of the bars OT' and OX. The bars OT' and OX' are extended to the left of the pivot O and are connected by means of the links s_1B_1 and s_2B_1, to form a rhombus. The bisector bar, B_1OB, passes through a guide at the point B_1 of the rhombus, so that OB is constrained always to remain at the same angular distance from both axes of either set.

[4] See, for example, O. W. Richardson, *The Electron Theory of Matter*, Cambridge, p. 336; 1916.

[5] A popular description of this model was published in "Science and Invention," *11*, p. 442; 1923.

The point A (Fig. 3) is represented in Fig. 12 by the center A of a pivot supported by the guide b from the bar OB. The bars a_1 and a_2, pivoted at A, can be used to represent the coordinates AC and AC' in Fig. 3. The bars V_1 and V_2 are provided for velocity measurements, in accordance with Fig. 7. At their centers they have pins which fit into the holes F and F'. The lower ends of the bars are fitted with guides

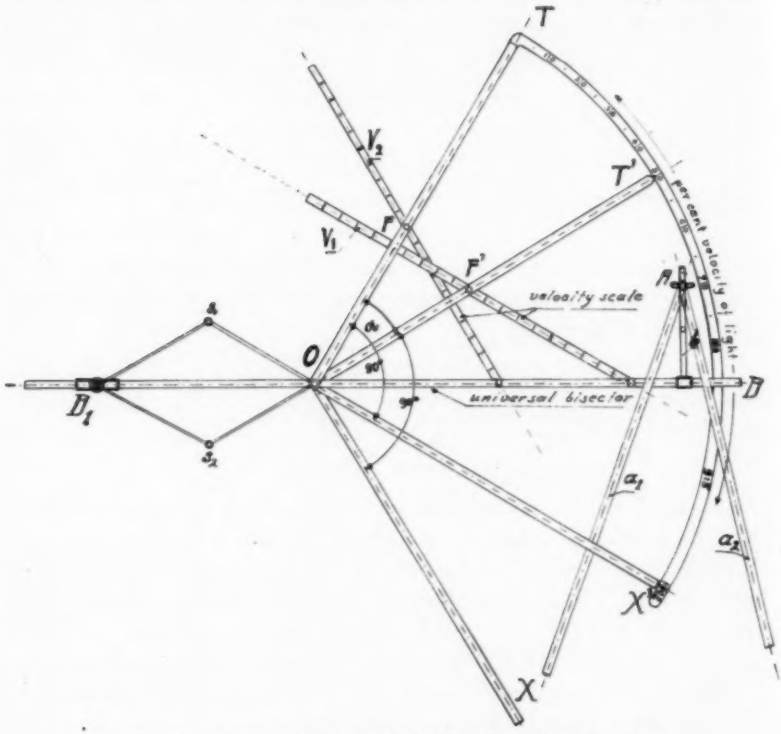

FIG. 12. *A mechanical model which illustrates the relations in the Lorentzian plane.*

which allow these bars to move freely along OB, when the angle α is varied. The scales on the bars V_1 and V_2 are graduated directly in decimal fractions of the velocity of light. On the bar OT is attached a circular scale which can be graduated in per cent of velocity of light, or in degrees, or in both.

By inverting the part b down and by adding another hanger at B_1, one of the bars, a_1 or a_2, may be suspended parallel to OB. The device

then directly shows the Doppler effect at different velocities. Other attachments can be added to illustrate various relationships described in the text; the above suggestions will be sufficient to indicate the construction and the mode of use of the device. It is convenient to paint one set of coordinate bars, say XOT, and the corresponding bars V_1 and a_1, red. Then everything that refers to the S observer is directly

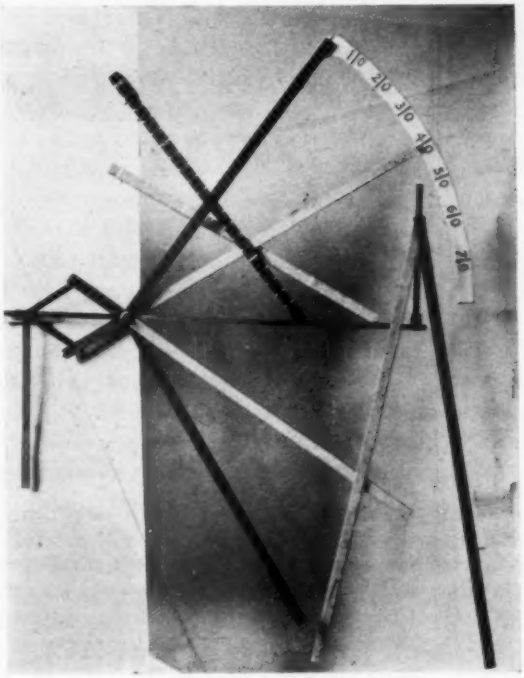

FIG. 13. *A photograph of the device shown in Fig. 12.*

distinguishable to the eye. Similarly, all the parts referring to the S' system can be painted say blue. The parts common to both systems, like the bisector OB, can be painted in some third color.[6]

[6] This model and the underlying theory have been generalized by the present writer to represent the astronomical aberration of light from the point of view of relativity. This phenomenon requires a two-dimensional space and consequently a three-dimensional model. See the *Journal* of the Optical Society of America, *9*, p. 223; 1924.

APPENDIX—A LEMMA

Some of the relations deduced in the text above are of the general form

$$\frac{m}{n} = \frac{\sin(a+b)}{\sin(a-b)} \tag{48}$$

For some purposes it is convenient to transform this relationship into Eq. (49). Namely, we have

$$\frac{m}{n} = \frac{\sin a \cos b + \cos a \sin b}{\sin a \cos b - \cos a \sin b}$$

By composition and division, we get

$$\frac{m+n}{m-n} = \frac{2 \sin a \cos b}{2 \cos a \sin b}$$

or

$$\frac{m+n}{m-n} = \frac{\tan a}{\tan b} \tag{49}$$

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Motion of Electrons in Oxygen.—Experiments made according to Townsend's method with purer oxygen than had previously been used, traversed by electrons liberated by ultraviolet light from a metal plate, showed that the various characteristic features of the current depended only on the ratio of electric field strength Z to pressure p, not on Z nor on p separately; whence the inference is that the electrons do not adhere to oxygen molecules but remain free over the whole range of pressures used (0.94 to 6.42 mm Hg). Earlier discordant results are ascribed to adhesion of electrons to molecules of impurities. The average kinetic energy of agitation $\overline{U}$ of the electrons is much in excess of that of the atoms, rising when $Z/p = 50$ to values exceeding 150 times that corresponding to a temperature of 15°C. The mean free path of the electrons, deduced according to the usual formula from the value of the above-mentioned quantity $\overline{U}$ and from their measured mobility, varies with $\overline{U}$; it has a maximum when $\overline{U}$ corresponds to 0.48 volt and a minimum when $\overline{U}$ amounts to 2.7 volts.—[H. L. Brose (Oxford); Phil. Mag., *50*, pp. 536–546; 1925.]

KARL K. DARROW

COMPARISON OF THE RED CADMIUM LINE IN THE VACUUM ARC AND IN THE DISCHARGE TUBE

By Frederick L. Brown

Some years ago the author made interferometer determinations of about a hundred lines in the spectrum of titanium, using as a comparison source a cadmium arc in vacuo.[1] At once the question was raised whether the wave length of the line given by the arc was the same as that given by a discharge tube; or in short, whether the vacuum arc could be used in place of the Michelson tube for precision work. Dr. Keiven Burns suggested the question would be worth investigating, using the same spectroscope and source used in the previous work. The opportunity to make this test came in 1924, when Prof. Henry Crew kindly invited the author to spend the summer at Northwestern University and placed the apparatus at his disposal.

The vacuum arc has already been described in detail in the original article.[2] It consisted, briefly, of a water cooled brass can provided with glass windows and an outlet for the air pump. The arc was carried by the conical cover of the can and when the cover was in place the arc was centrally located in the can. Attachments were provided for raising and lowering the electrodes, etc. When in operation the pressure was kept below 10 mm of Hg. The arc drawing from 1 to 3 amperes was maintained by a 220 volt D. C. supply.

The cadmium tubes used in this work were fused quartz H-tubes provided with aluminum electrodes. These tubes were carefully wrapped with asbestos paper and about 50 ohms of No. 26 heating wire; a small window being left opposite the capillary of the H-tube. When in use the tubes were attached to a vacuum pump, and in general the pressure was between 2 and 4 mm of Hg. Cadmium in the form of filings was introduced before the tube was connected to the pump. The heating current was controlled by resistances so that the tube should not be injured; in general it required about 1 ampere to keep it at the correct temperature. The current for exciting the spectrum was provided by a 5000 volt transformer working on a single phase 60 cycle 110 volt A. C. line with a primary current of from 1 to 2 amperes.

[1] Brown, Astrophysical Journal, *56*, pp. 53-70; July, 1922.
[2] Brown, Loc. cit.

In addition to the cadmium sources a neon tube, and a Cooper Hewitt quartz mercury lamp were used for comparison and control purposes. The neon tube could be connected to the 5000 volt transformer. The Hg arc was operated with 110 volt D. C. supply and gave sharpest rings when run at the minimum possible current, about 2 amperes.

The sources were compared by means of a Fabry-Perot interferometer using in all four quartz etalons of approximately 5, 10, 20, and 30 mm length. Dr. Geo. S. Monk of the University of Chicago very kindly resilvered the plates of the interferometer with the cathode sputtering apparatus used in his work. The films were excellent and gave very good rings.

The optical system used is shown in Fig. 1. *A* is either the Cd arc, the Hg arc or the Ne tube. *B* is a lens focusing the light through the diaphragm, *D* on the etalon, *E*. The ring system was projected onto the slit of the collimator by the doublet, *F*; this lens is much better

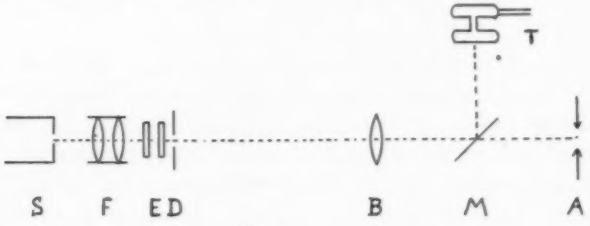

FIG. 1

than the one used in the previous work. The spectroscope, *S*, is a Hilger constant deviation glass prism type. The photographs were taken on W and W panchromatic plates, hypersensitized with ammonia. The light from the Cd-tube, *T*, was thrown onto the lens, *B*, by means of a movable mirror, *M*. This mirror was of course swung out of the way when photographing sources at *A*. The fixed positions of the lens *B*, and the diaphragm, *D*, made it certain that the optical paths would be similar. The image of the Cd H-tube controlled the size of the diaphragm, *D*, used in front of the etalon, as it seemed advisable to use the same area of the interferometer plates for all sources compared. The tube required exposures of from 30 minutes to one hour. An effort to reduce the time by increasing the exciting current or the heating current and so increase the brilliancy of the tube indicated clearly the danger there was of spoiling the tube, principly by softening the aluminum electrodes. Owing to the small diaphragm the Ne-tube exposures were also long, ranging from 20 to 40 minutes; but the Hg and Cd arcs were

so much more intense that they required only a few minutes. The long exposures required by the tubes when using full illumination made it seem inadvisable to use the method of simultaneous exposure with a half silvered mirror at M. Instead a series of exposures was taken of the different sources in such an order as to most easily record any change in the ring system due, say, to temperature change. The question of a thermostat for the etalon was considered, but in view of the short time available, for the work, it seemed best to get along without it rather than run the risk of not securing the photographs at all. This seemed justifiable from the fact that the summer of 1924 was very cool in Evanston, and the further fact that the etalons were made of quartz. Of course this omission makes the temperature effect the greatest source of error, and it will be discussed in detail later.

The $3\frac{1}{4}$ by $4\frac{1}{4}$ plates used in the spectrograph make possible three different sets of exposures on each plate. The order of the exposures on the 20 mm plates was, 1st—Ne tube and Cd arc; 2nd, Cd tube; 3rd, Ne tube and Cd arc. On the 30 mm plates the order was, 1st, Ne tube, Hg arc, Cd Arc; 2nd, Hg arc, Cd tube; 3rd, Hg arc, Cd arc, Ne tube. These two sets are by far the best from every point of view. The plates taken with the 10 mm etalon did not show up well under the measuring machine, so that only two admit of careful measurement. The plates taken with the 5 mm etalon have no Ne spectrum as the Ne tubes on hand had either cracked or had become too hard, and a new one ordered did not arrive until after the author had left for Virginia. While waiting for the new Ne-tube some pictures were taken with the 5 mm etalon using only Hg as comparison, the order of exposures on these was: 1st, Hg arc and Cd tube; 2nd, Hg arc and Cd arc; 3rd, Hg arc. The plates were all brought to Virginia and have been measured and reduced here.

In order to measure all the rings of all the lines without touching the plates and changing the orientation, a Gaertner comparator was mounted so as to move a low power microscope at right angles to the main carriage of a Socíeté Genevoise measuring engine. The plates were carried on the main carriage of the S. G. machine, and all rings measured in terms of the screw of this machine which gives their diameters directly in millimeters.

Before outlining the method of computation which while not new has been developed especially for the present work it will be well to state definitely what it was desired to do. The primary purpose was to compare the wave length of the red Cd line as given by a discharge

tube and by a vacuum arc; the difference, if any, would be small. Everything else remaining constant, the change in wave length would produce only small changes in the diameters of the interference rings. In the second place, as there was no accurate temperature control, it was essential to look for changes produced by variations in the length of the etalon. Here again the effect if present would show up as small changes in the diameters of the rings. In the third place, there might be apparent changes due to the different intensities of the Cd sources; this again would result, if present, in small differences in the measured diameters of otherwise identical ring systems. A method of reduction was therefore needed which would consider these small changes in as accurate a manner as possible; it need not involve the reduction of any of the systems to actual wave lengths. With this in mind the reasons for the method chosen are obvious.

The standard form given by Pfund[3] and others for the computed wave length λ is

$$\lambda = \frac{2e}{p}\left(1 - \frac{D_n^2}{8R^2}\right) \qquad (1)$$

where e is the distance between the interferometer plates

D^n is the diameter of the nth ring from the center

p is the order of interference for this nth ring, and

R is the effective focal length of the photographing system.

Furthermore, the difference in the squares of consecutive rings is a constant

$$c = D_n^2 - D_{n-1}^2 = \frac{8R^2\lambda}{2e}$$

so that these squares form an arithmetical progression.

The variation in the computed wave length with changes in D_n^2 becomes by differentiation,

$$d\lambda = -\frac{2e}{8pR^2} d(D_n^2)$$

Let $a = d(D_n^2)$ be the change in the square of the measured diameter. Then the corresponding fractional change in the wave length is

$$\frac{d\lambda}{\lambda} = -\frac{2e}{8p\lambda R^2} a$$

[3] Pfund, Astrophysical Journal, *28*, pp. 197-211; 1908.

But $2e/p\lambda$ is very nearly unity as shown by equation (1) so that the fractional change becomes $(d\lambda/\lambda = -a/8R^2)$. Since R is very nearly constant over the visual region and for all the etalons, the fractional change in the wave length will depend only on a and not at all on λ or e.

Suppose now the squares of the diameters of corresponding rings on two exposures of the same line on the same plate be compared, and it is found that $(D_1^2 - D_2^2 = a)$ is not zero. Neglecting for the moment accidental errors, this difference might mean any one of three things: 1st, that the wave length of the source changed between exposures; 2nd, the etalon had changed length between exposures, due say to change of temperature; 3rd, that the intensities of the corresponding rings were so different that due to unsymmetrical broadening in the case of the darker one, the measured diameter was systematically larger or smaller, than the measured diameter of the fainter one. But in any case regardless of the cause of the difference the wave lengths computed differ by, a parts in $8R^2$.

By using a large number of rings the accidental errors of measurement should to a great extent average out so that the mean value of, a, given by the comparison of from six to twelve pairs of corresponding rings should be fairly good; and would in the absence of the causes mentioned above, be very nearly zero. If, however, temperature increases the length of the etalon between two exposures of the same source on the same plate, then the rings of the second exposure will all be slightly larger than they were before, and $(D_1^2 - D_2^2 = a)$ will be negative. And as it will increase the etalon distance in the same proportion for all wave lengths, the value of a (due to this cause alone) will be the same for all the lines. The neon tube and mercury arc were sources easy to control and of constant brightness, so that it was a simple matter to secure two exposures having the same intensity; and as the wave lengths would be constant, the only valid reason for a being different from zero in the case of these lines, aside from accidental errors, was a change in etalon length due to a change in temperature. All the plates taken show such a change, though in general it is quite small; sometimes it is in one direction sometimes in the other. In making this comparison between the exposures, five neon lines 6506, 6402, 6266, 6143 and 5852, and two mercury lines, 5790 and 5769 were used. On some of the plates the neon line 6266 is too faint to give good rings. The two yellow lines of mercury give very sharp ring systems even in the 30 mm etalon, but the other lines did not give good ring systems for the higher orders.

Suppose now that the Cd arc and Cd tube exposures are compared. There will be some change due to the temperature effect, but this can be corrected for, if we assume that the change is a linear function of the time, and use the neon and mercury lines to determine its amount. Any remaining difference is then due either to different wave lengths in the two sources, or to intensity effects. It is very difficult to gauge the exposures for these sources, especially the arc, so as to give equal intensities; and even if this is attained for the red line which is of primary importance it will not hold for the other Cd lines as in the tube the red line is the brightest, and in the arc the faintest of the four. The tube apparently gives a slightly sharper red line, though the arc is much more intense and requires a much shorter exposure. This has also been indicated by Monk.[4] Consequently it is difficult to say off hand how much of the difference between the two ring system is due to different wave lengths, and how much to intentisy effect. If one assumes that the intensity effect varies directly with the difference in intensities then one can write for each pair of arc-tube values an equation of the form: $(X+b(I_1-I_2)=a')$.

where X is the difference due to change in wave length,

b is the intensity coefficient

I_1 and I_2 the intensities in the Arc and in the Tube respectively on a basis of 0 to 20 as used in this work, and

a' is the difference between the two rings systems corrected for temperature, that is $a'=a+k$ where k is the temperature correction determined from the neon lines, the mercury lines and the comparison of the first and last Cd arc exposure lines, with special weight given of course to the neon and mercury lines.

As each plate will in general have two exposures of the Cd arc and one of the Cd tube each plate will in general give two equations. If the intensities of the two arc exposures are different the two constants, X and b, could be determined from a single plate. But as the method of reduction admits of comparing several plates the values have been determined by least squares solution of groups of plates. The values of X, and b, as well as the data from which they are computed are all collected in Table 1. A glance at this table shows that the values of X and b do not have the same sign nor the same magnitude for the various groups of plates, also that the residuals are rather large, so that one feels that they do not give any definite quantitative determination.

[4] Monk, Chicago Meeting of the Physical Society, Nov. 1925.

TABLE 1

DATA FOR THE CADMUM LINE 6438.4696

Plate No.	Etalon Length	a	k	a'	I_A	I_T	Residuals
140	4.9mm	− 90	−12	−102	10	7	−43
142	"	−149	+28	−121	10	5	− 3 X = +30.5
143	"	− 11	− 1	− 12	10	7	+47 b = −29.8
144	"	+ 38	− 9	+ 29	10	10	− 2
		Mean		− 52			24
109	9.8	+ 14	+15	+29	8	5	+21
		+ 61	−12	+49	12	5	+ 3 X = −19.6
110	"	−115	+29	−86	3	5	−48 b = + 9.36
		−104	+71	−33	3	7	+24
		Mean		−10			26
115	19.9	+23	+ 6	+29	8	12	+25
		+18	− 4	+14	13	12	− 2
116	"	−16	+ 3	−13	9	14	−15 X = +13.5
		+24	− 5	+19	12	14	+10
117	"	−11	+ 9	− 2	12	15	− 8 b = +2.43
		+11	− 8	+ 3	13	15	− 6
118	"	+ 3	− 1	+ 2	12	15	− 4
		Mean		+ 7			10
130	29.9	+ 2	+ 5	+ 7	5	10	+ 6
		− 9	− 5	−14	12	10	− 7
131	"	+67	−73	− 6	15	9	+ 6 X = −4.8
		−90	+77	−13	13	9	− 3 b = −1.24
132	"	+16	−13	+ 3	6	10	+ 3
		−20	+13	− 7	6	10	− 7
133	"	+62	−41	+21	8	8	− 2
		−13	+41	+28	10	8	+ 7 X = +22.9
134	"	+24	−14	+10	13	9	− 9 b = − 1.05
		+ 6	+14	+20	15	9	+ 3
Mean of all five plates				+ 5			5

a is the average difference between pairs of corresponding rings, taken in the sense Cd-arc minus Cd-tube.

k is the temperature correction.

a' is a corrected for temperature change.

I_a and I_t are the intensities of the arc and tube exposures on a scale of 1 to 20. Residuals are given in the sense Computed minus Observed, and are computed from the values of X and b given for the several groups in the last column. The values of a, k, a^1, and Res. are given in terms of 0.01 mm² as a unit. For wave length 6438, 0.01 mm² equals 0.000080 A. U.

But they do show that the intensity plays an important part in the comparison especially if the difference in intensity is large. The recent article[5] by Keiven Burns in the J.O.S.A. on the red neon lines emphasises this.

As this intensity error had been pointed out by Sparrow[6] and others before the work was begun, an effort was made to determine it independently of any other effects. To do this two plates were taken having in all six exposures of the mercury lines with the 5 mm etalon. The exposures range from 20 seconds to 27 minutes and are taken in such an order as to make correction for temperature effects, although these are negligibly small. These plates show a decided intensity effect, but of course its value cannot be used on the other plates for as Sparrow has shown[7] the intensity pattern may change with orientation or with any other changes in the positions of the two interferometer plates. Certainly the effect is not constant if the values of b, determined for the several groups of plates give any true representation of it.

Since the effect cannot be determined, and seems to vary in sign as well as in magnitude it may be assumed to more or less average out in taking account of a large number of exposures where the intensities differ in amount with sometimes the arc the more intense, sometimes the tube. Allowing this assumption a direct comparison of arc and tube can be made by averaging the several values of a' given in the table. Averaged by sets, we find that the two short etalon sets give negative values for a', and the two long ones give positive values for a': and a general average giving double weight to each of the individual long etalon observations indicates that the Cd arc line 6438 is longer than the Cd tube line by $a' = -1$ or about 0.0001 A. U. For the other lines the average difference is larger and the available data less plentiful so that only the final results are given here. If the accepted values be assigned to the radiation from the tube we have the following table:

Tube	Arc
6438.4696	6438.4697
5085.822	5085.823
4799.911	4799.913
4678.152	4678.151

Of course, if still more weight be given to the long etalon results, on the basis that they are by far the more accurate, then the arc lines may

[5] Burns, J.O.S.A. & R.S.I., *11*, p. 309; Oct. 1925.
[6] Sparrow, Astrophysical Journal, *49*, pp. 90-95; March, 1919.
[7] Sparrow, Loc. cit.

all be found to be shorter than the tube lines. But however they may be weighted to form an average, it seems certain that the arc and tube do not differ, for the red Cd line, at least, by as much as 0.001 A. U. one way or the other. So that the Vacuum arc can be used as a safe source for all but the most precise work at any rate, and will probably be found all right even for this if the current density is not too great, and the pressure is kept below 10 mm of Hg. Certainly its greater intensity will make it valuable if this point is given further verification.[8]

The author realizes more than ever the need for a temperature control for this work, and probably in spite of the attempted correction it still affects the results to some degree, but the fact that it is sometimes positive and sometimes negative, makes him hopeful that it has averaged out at least sufficiently to give the work some real value, as a comparison between these two sources of radiation.

One other item of observation which, though not directly on the point in hand, may be of sufficient interest to admit of inclusion relates to the behavior of the Cd discharge tube itself. When the tube was observed with a direct vision spectroscope while being exhausted and heated the electrodes having been already connected to the high voltage supply, the following changes in the spectra were noted. As soon as the pressure was low enough the ordinary nitrogen spectrum due to the residual air became visible. As the tube was heated the positive bands in the red and orange became very brilliant, the Cd lines were still invisible. Then suddenly the Cd lines appeared and grew rapidly in intensity, the nitrogen positive bands fading away just as quickly so that in a few seconds only the Cd lines and possibly a few of the violet nitrogen bands were left. If heated much above this point the green Thallium line appeared and the aluminum electrodes became soft.

By careful heating it was possible to keep the positive band system in view for a long time. Two pictures, without the etalon, were taken of the tube in this condition and showed the fine structure of the positive bands as well as the dispersion of the instrument would be expected to warrant.

The suddenness of the change from positive bands to Cd-lines on heating and the corresponding change in the reverse direction on cooling were very marked and were beautiful to observe. The author is in doubt whether differences in the ionization potential of Cd and N, or

[8] Monk, Loc. cit.

collisions of the second kind best explain the observations. It seems reasonable to suppose that the change takes place on heating when the Cd begins to evaporate, and on cooling when it condenses, but this does not explain it.

In conclusion, the author wishes to express his appreciation to Prof. Crew for making the investigation possible, and for his interest and encouragement during its progress. Acknowledgements are also due to the late Mr. Küng, who as mechanician at Northwestern University, assisted in making parts of the apparatus. Also acknowledgment is gratefully made to other members of the staffs at Northwestern University and the University of Virginia who have helped the work to completion by suggestions and encouragement.

UNIVERSITY OF VIRGINIA,
UNIVERSITY, VA.,
DECEMBER, 1925.

X-rays. By Maurice de Broglie. Translated from the French by J. R. Clarke. xiii+204 pages. E. P. Dutton and Company, New York. $5.00.

This volume first appeared in French in 1922, and because of its terse, yet complete, treatment of the subject, is well known to most physicists working in x-rays and allied fields. The present translation, however, is really a second and revised edition, since it contains much work that has appeared since 1922. For example there is a discussion of the Compton effect; of the refraction of x-rays, containing references as late as April, 1925; and of recent work in β-ray and γ-ray spectra.

The content of the book is best indicated by giving the chapter headings: Bohr's Theory; The Scattering of X-rays by Matter; The Absorption of X-rays by Matter; The Continuous Spectrum; Line Spectra (including valuable tables of wave lengths); Spectrographs and Spectrometers and Sources of X-rays; The Refraction and Total Reflection of X-rays. Each chapter is followed by a valuable list of references.

The book is as up to date as is possible for a book to be in this rapidly developing field. It will serve not only as a valuable summary, but, with the carefully selected references, it will provide a most excellent outline to form the basis of a course in x-rays for graduate students.

F. K. RICHTMYER

TABLE 2. *The Computational Importance of the Collinearity Condition**

p	x'	$3x'$	$1-3x'$	$\frac{f'x=}{\frac{1-3x'}{1-3X}}$	$\delta f'x=$ $fx'-fr'$	$\frac{\delta x'=}{\frac{(1-3X)\delta fx}{3}}$	$x=$ $x'+\delta x'$	$3x$	$1-3x$	$\frac{1-3x}{1-3X}$	xL_x	ΣxL_x	$\frac{x}{\Sigma xL_x}$	$1-\frac{x}{\Sigma xL_x}$	p	p'
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
	R .000	.000	1.000	1.00			.0000	.000	1.000	1.00	.0000		.0000	1.0000		
1.00	G .007	.021	.979	1.00			.0070	.021	.979	1.00	.0043	.0162	.4320	.5680		
	B .993	2.979	1.979	1.00			.9930	2.979	1.979	1.00	.0119		61.30	60.3		
	r .006	.018	.982	.982	.000	.0000	.0060	.0180	.9820	.982	.0022		.2727	.7273	.727	.741
.80	g .015	.045	.955	.976	.006	.0020	.0130	.0390	.9610	.982	.0080	.0220	.5909	.4091	.721	.622
	b .979	2.937	1.937	.979	.003	.0020	.9810	2.9430	1.9430	.983	.0118		44.59	43.59	.723	.683
	r .015	.045	.955	.955	.000	.0000	.0150	.0450	.9550	.955	.0055		.4933	.5067	.507	.521
.60	g .023	.069	.931	.951	.004	.0013	.0217	.0651	.9349	.955	.0133	.0304	.7119	.2881	.507	.467
	b .962	2.886	1.886	.953	.002	.0013	.9633	2.8899	1.8899	.955	.0116		31.69	30.69	.508	.493
	r .034	.102	.898	.898	.000	.0000	.0340	.1020	.8980	.898	.0126		.6995	.3005	.301	.298
.40	g .040	.120	.880	.899	.001	.0003	.0403	.1209	.8791	.898	.0249	.0486	.8290	.1710	.301	.306
	b .926	2.778	1.778	.898	.000	.0003	.9257	2.7771	1.7771	.898	.0111		19.03	18.03	.299	.301
	r .074	.222	.778	.778	.000	.0000	.0740	.2220	.7780	.778	.0274		.8544	.1456	.146	.154
.20	g .081	.243	.757	.773	.005	.0016	.0794	.2382	.7618	.778	.0490	.0866	.9170	.0830	.146	.131
	b .845	2.535	1.535	.776	.002	.0016	.8466	2.5398	1.5398	.778	.0102		9.780	8.780	.146	.144
	r .130	.390	.610	.610	.000	.0000	.1300	.3900	.6100	.610	.0481		.9303	.0697	.070	.053
.10	g .130	.390	.610	.623	.013	.0042	.1342	.4026	.5974	.610	.0828	.1397	.9606	.0394	.069	.092
	b .740	2.220	1.220	.616	.006	.0042	.7358	2.2074	1.2074	.610	.0088		5.268	4.268	.071	.073

$$p=\frac{1-\frac{x}{\Sigma xL_x}}{1-\frac{X}{\Sigma XL_x}},$$

where $\Sigma xL_x=rL_r+gL_g+bL_b$,
$\Sigma XL_x=RL_r+GL_g+BL_b$.
$L_r=0.370$,
$L_g=0.617$,
$L_b=0.012$.

* Columns 1 and 2 were taken from Weaver's data* and represent (1) purity which is one minus one-hundredth the per cent white given in Weaver's table; and (2) the trilinear coordinates (r, g, b) of points on the 440 mμ wave length line. The point corresponding to unit purity is designated as (R, G, B) according to the present notation. The column is headed, x', which may represent either r, g, or b, as indicated. The g-coordinate, not given in Weaver's table, has been calculated from the relation: $g=1-r-b$; which imposes the second condition, (see introduction).

Column 5 is the test for wave length constancy (Condition 4), and the variations within triads show that the condition is not quite met, or $(1-3r)/(1-3R)$ does not quite equal $(1-3g)/(1-3G)$ or $(1-3b)/(1-3B)$.

Each parameter, therefore, defines a distinct point on the 440 mμ wave length line, the point indicated by r being different from that defined by g or by b. It is planned, however, to alter g and b slightly so that they correspond to the point indicated by r, and to show that this slight alteration (in general less than 0.002, see Column 7) is enough to make the calculated purities check.

Column 8 gives the trilinear coordinates of the r-point correct to four places as is shown by the constancy of the triads in column 11.

Column 16 gives the values of purity by all three of the Tuckerman forms, as the headings indicate. The deviations of these purities from those (Column 1) indicated in Weaver's table are quite marked and have already been commented upon by Priest.**

Column 17 gives the purities from the same formulas but by applying them "blindly" to Weaver's uncorrected data (Column 2). These are the same values as are plotted on Priest's comparison graph.***

The very material reduction of the variation within the triads of Column 16 as compared to that in Column 17 shows the computational importance of the fourth equation of condition (condition for collinearity with spectral point and white point).

* J.O.S.A. & R.S.I., *6*, No. 6, p. 586; August 1922. ** Priest: Loc. cit. p. 517. *** Priest: Loc. cit., p. 517.

THE VALIDITY OF FLICKER PHOTOMETER MEASUREMENTS IN HETEROCHROMATIC PHOTOMETRY

By A. H. Taylor

The use of the flicker photometer for comparing lights which differ greatly in color has been the subject of many researches. This instrument has many points in its favor, but the validity of the results obtained has not been established to the satisfaction of everyone. Under some conditions it has been shown that flicker and "direct comparison" photometers, such as the Lummer-Brodhun contrast photometer, give consistently different results when there is an appreciable color difference. However, there has been little or no proof that *either* method correctly evaluates differently colored lights as regards their usefulness or luminosity.

In 1912 Ives[1] published the results of an extensive study of photometric measurements by the two types of photometers, and specified the conditions under which they would give substantially the same results. He subsequently[2] developed a method for determining the color sensitivity of individual observers so that results by several observers could be correlated. From observations by several observers the value which would have been obtained by an observer of "average" or "normal" color vision could be determined. This method of procedure was further studied by Crittenden and Richtmyer[3] at the Bureau of Standards. The apparatus and experimental procedure are as follows:

The apparatus used includes a bar photometer, Kingsbury[4] type of flicker photometer, two carbon lamps operating at 4 watts per candle and two solution cells with colorless glass sides. One cell is filled with a solution of potassium bichromate in distilled water, made up in the proportion of 72 grams to one liter of solution. The other cell is filled with a copper sulphate solution, 57 grams to one liter of solution. Light from the comparison lamp passes through a one-centimeter layer of one of these solutions. The observer determines the integral transmis-

[1] Ives, Phil. Mag., p. 149; July, 1912.
[2] Ives, I. E. S. Trans., p. 315; 1915.
[3] Crittenden and Richtmyer, I.E.S. Trans., p. 331; 1916.
[4] Kingsbury, Jour. Fr. Inst., *180*, p. 215; August, 1915.

sion-factor of each solution for light from the carbon lamps operated at 4 watts per candle. If Y and B are the transmission-factors obtained for the potassium bichromate and copper sulphate solutions, respectively, then the observer's color-vision characteristic for the purposes of heterochromatic photometry is defined by his value of the ratio Y/B. For the "average" or "normal" observer this ratio is 1.0.

In order to use observers whose values for Y/B are different from 1.0 their photometric results obtained in flicker-photometer observations involving any degree of color difference are plotted against their values of Y/B, as shown in Figs. 1 to 3, which will be explained later.

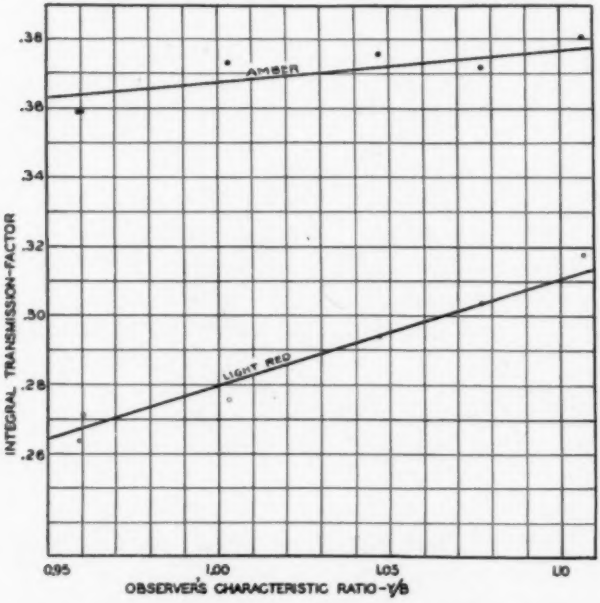

FIG. 1. *Flicker photometer measurements of transmission-factor of amber and light-red color filters, showing effect of observer's characteristic color ratio, Y/B, upon the value obtained.*

In general a straight line best represents the results of a group of observers, and from it can be scaled off the value for a Y/B ratio of 1.0, the value which would be obtained by a "normal" observer.

The above procedure is the only one which has reached any degree of standardization, and hence it is important to know whether results obtained by this method are valid, or agree with results by any other reliable method.

Ferree and Rand[5] have attacked the validity of flicker photometer observations because of the difference in rate of rise and decay of luminosity sensations with differently colored lights. Their tests showed that the luminosity value assigned to any colored light was dependent upon the color and brightness of the photometric field and the duration of exposure. Their conditions of measurement were entirely different from those encountered in the "standardized" method of flicker photo-

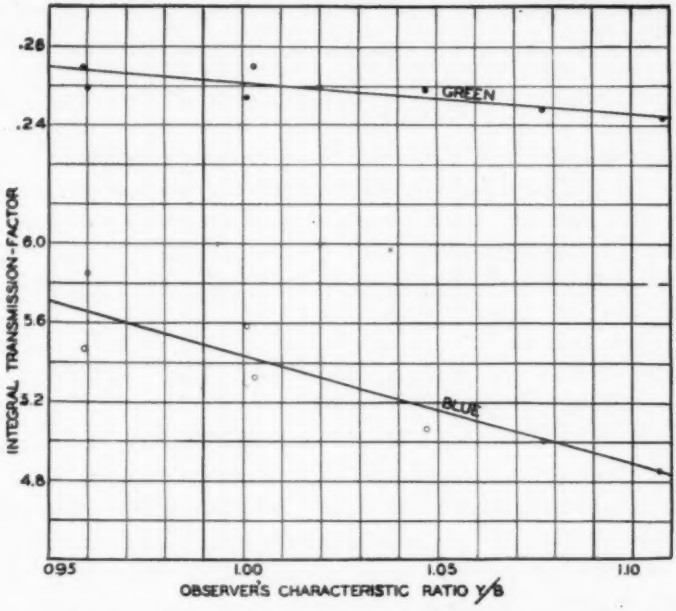

FIG. 2. *Flicker photometer measurements of transmission-factor of green and blue color filters*

metry described above, and hence it is questionable whether their conclusions are applicable to measurements made by this method.

The author has recently determined the integral transmission-factors of thirteen color-filters for use in routine photometric tests in the laboratory. These filters included four red glasses and three each of amber, green and blue. All of them were sufficiently saturated in color to make it very difficult if not practically impossible to photometer them by the ordinary direct-comparison photometer methods.

Six observers, four of whom had never used a flicker photometer previously, and had had little or no photometric experience, were used

[5] Ferree and Rand, Trans. I.E.S., p. 151; Feb. 1923.

for the tests. In one or two cases a seventh inexperienced observer was used. Their characteristic Y/B ratios were first determined; then each observer made a series of measurements on each color filter. The lamp used with the color filters was operated at a color-temperature of 2680°K, as determined by careful measurements by the Nela Research Laboratory. The illumination of the photometer screen was approximately 50 meter-candles for most of the measurements. When making an observation the photometer and the lamp on the left were stationary, and the

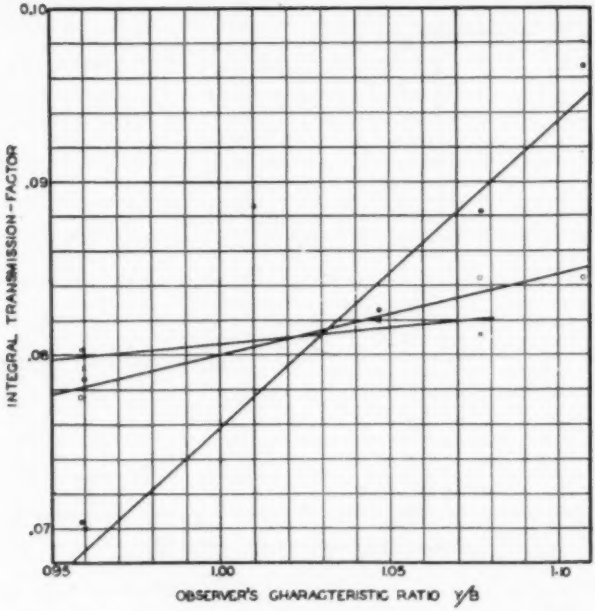

FIG. 3. *Curves (a) and (b)-flicker photometer measurements of transmission-factor of a deep ruby color filter, measured in terms of two standardized light-red filters. Curve (c)-transmission-factors of same ruby glass, when color difference was taken in one step.*

comparison lamp on the right was moved until the flicker disappeared or was a minimum. The color filter being measured was placed between the photometer and the comparison lamp, and an unstandardized filter of intermediate color was kept on the left-hand side throughout the measurements of all glasses of the same color. This was done to divide the color difference into two parts, but probably did not in any way influence the final results. In each case the rate of flicker was

adjusted to the minimum value for which it was possible to make it disappear, or at which a judgment of minimum flicker could be made. This required a considerably higher speed for red glasses than for blue ones.

In the measurement of one of the characteristic solutions the effect of rate of flicker was tested. Observations were made at the minimum flicker rate at which they could be made, and also at a rate 50 per cent higher. The difference in results in the two cases was below 1 per cent, and of the same order as the experimental error of two sets of observations under identical conditions. (See also bibliography.[6])

Figs. 1–3 show the results of flicker photometer measurements of transmission-factors of five representative color-filters. The value by each observer is plotted against his value of Y/B, and the straight line best representing the results is drawn. From this line the value for the normal observer ($Y/B = 1.0$) is obtained. The ruby glass (Fig. 3) was measured in two ways. Photometric readings were taken with a standardized lighter red glass on the left and the ruby glass on the right-hand side of the photometer; then the two filters were interchanged and more readings were taken. This was done with two standardized red glasses, and the results are shown in curves (*a*) and (*b*). This method of interchanging the filters reduces the photometric error one-half, and also greatly reduces the color difference encountered. Curve (*c*) shows the results of values obtained when readings were taken with and without the ruby glass on the right-hand side, an intermediate red glass of unknown value being kept on the left-hand side of the photometer throughout the measurements. In this case the color difference was very great, and measurements were extremely difficult, even with the flicker photometer. The flicker speed was quite high, which reduced the accuracy by increasing the brightness range over which there was little change in flicker. In color differences as great as this it is better to divide the measurement into two or more steps.

Fig. 4 shows the spectral transmission curves for the same five color-filters.

It is also possible to compute the integral transmission if certain data are known.

[6] Taylor, Trans. I.E.S., p. 574; Dec. 30, 1921.

Let J_λ = Relative energy of light-source at wave length λ.
V_λ = Relative visibility of energy at wave length λ.
T_λ = Transmission-factor of light-filter at wave length λ.

If we assume that the effective limits of the visible spectrum are 0.44 μ and 0.72 μ, then the integral transmission-factor T of the light-

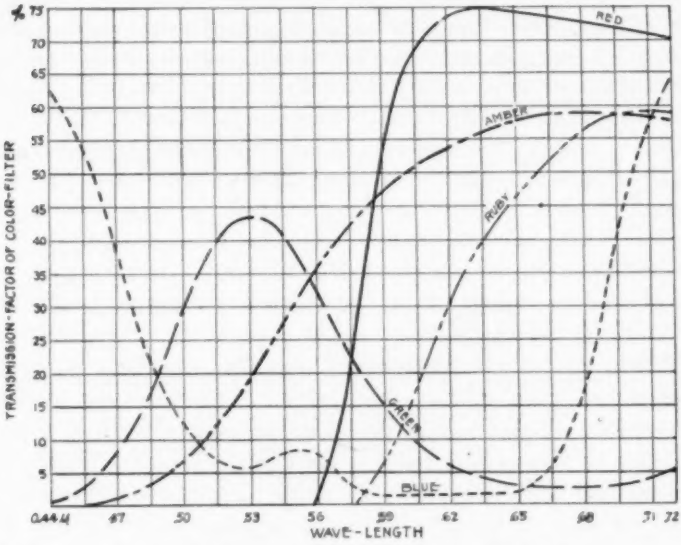

FIG. 4. *Spectral transmission curves for the same five color filters.*

filter for the particular light-source considered is given by the equation

$$T = \frac{\int_{0.44\mu}^{0.72\mu} J_\lambda V_\lambda T_\lambda d_\lambda}{\int_{0.44\mu}^{0.72\mu} J_\lambda V_\lambda d_\lambda}$$

By the use of the spectral-energy charts given in Bureau of Standards Miscellaneous Publications No. 56 the relative energy values of the tungsten lamp used in these tests, at color-temperature 2680° K, have been determined for various wave lengths.

In Table 1 are given numerical values of the various factors necessary for an evaluation of the above equation for the lamp and light-filters used in this test.

From the data in Table 1 the integral transmission-factors of the five glasses have been computed, using wave length increments of 0.01μ. In addition, computations have been made using average visibility values[7] obtained in the researches of Coblentz and Emerson, Hyde, Forsythe and Cady, and Gibson and Tyndall, as these are the three most extensive investigations in number of observers used and spectral range covered.

TABLE 1. *Data on relative-energy J_λ of tungsten lamp at color temperature of 2680°K, relative visibility V_λ of radiant energy and transmission-factors T_λ of color-filters at various wave lengths λ.*

λ	J_λ for 2680°K	*V_λ	Transmission-factor of color-filter				
			Ruby	Red	Amber	Green	Blue
0.44μ	0.2462	0.023	0	0	0	0.003	0.632
.46	.3345	.060	0	0	0.004	.035	.498
.48	.4395	.139	0	0	.019	.136	.260
.50	.560	.323	0	0	.066	.296	.120
.52	.695	.710	0	0	.142	.418	.059
.54	.841	.954	0	0	.245	.429	.074
.56	1.000	.995	0	0.017	.350	.326	.078
.58	1.167	.870	0.017	.314	.436	.198	.022
.60	1.340	.631	.141	.657	.497	.104	.015
.62	1.517	.381	.310	.732	.534	.052	.016
.64	1.695	.175	.426	.746	.562	.033	.016
.66	1.872	.061	.504	.742	.581	.026	.036
.68	2.044	.017	.561	.731	.591	.026	.165
.70	2.217	.0041	.588	.715	.589	.031	.470
.72	2.385	.001	.589	.697	.577	.038	.635

* Values of Relative Visibility recently adopted by the American Engineering Standards Committee.

Table 2 shows the integral transmission-factors obtained by computation from the four sets of visibility data, compared with the values for a "normal" observer as determined by the flicker photometer.

Table 3 gives a comparison between the values as observed by the flicker photometer, for a normal V/B ratio of 1.0, and those computed by use of various visibility data.

It is seen from Table 3 that if we assume the visibility data adopted by the American Engineering Standards Committee to be correct, then the flicker photometer gives results which are low for the red and

[7] Gibson and Tyndall, B. S. Sci. Pap. No. 475.

amber filters, and high for green and blue. Stated another way, it appears that the flicker photometer undervalues the orange and red end of the spectrum, and overvalues the green and blue portions.

TABLE 2. *Transmission-factors of five color-filters as computed by use of four sets of spectral visibility data, and as observed by use of the flicker photometer.*

Visibility Data By	Transmission-Factors of Filters				
	Ruby	Light Red	Amber	Green	Blue
Coblentz and Emerson	10.1%	31.6%	38.3%	22.7%	4.88%
Hyde, Forsythe and Cady	8.8	29.0	37.2	23.8	5.09
Gibson and Tyndall	9.7	30.5	37.8	23.2	5.09
Amer. Eng. Standards Comm.	9.5	30.2	37.8	23.3	5.06
Observed by Flicker Photometer (for $V/B=1.0$)					
	*8.0	27.9	36.6	25.1	5.44
	**7.6				

* Color Difference Taken in two Steps.
** Color Difference Taken in one Step.

Assuming the equality-of-brightness photometer ratings of monochromatic light to be correct, Ferree and Rand[5] obtained just the opposite results. By their test methods they found that the flicker photometer, with a field illumination of approximately 50 meter-candles, gave high

TABLE 3. *Relative transmission-factors of five color-filters as observed by the flicker photometer and as computed from spectrophotometric data, spectral energy, values at color temperature 2680°K and various visibility data.*

Visibility Data By	Transmission-Factors Relative to Values Observed by Flicker Photometer				
	Ruby	Light Red	Amber	Green	Blue
Coblentz and Emerson	125.5	113.3	105.6	90.4	89.7
Hyde, Forsythe and Cady	109.3	103.9	101.6	94.8	93.6
Gibson and Tyndall	120.5	109.3	103.3	92.4	93.6
Amer. Eng. Standards Comm.	118	108.2	103.3	92.8	93.0
Observed by Flicker Photometer	100	100	100	100	100

values to light of wave lengths .675 and .579μ, and low values to light of wave lengths .515 and .466μ. It has already been stated in this paper that their test methods were entirely different from those used in the standardized flicker photometer procedure, hence it does not appear that their conclusions are applicable to the latter.

Gibson[8] has recently computed the transmission-factors of the Y and B solutions described above, using various visibility data. On the basis of the visibility data adopted by the American Engineering Standards Committee his computed value of the ratio Y/B is 1.1 per cent below the value obtained by the flicker photometer researches of Crittenden and Richtmyer.

While the differences between the computed and observed values in Tables 2 and 3 are appreciable, they are not excessive in view of the difficulties involved in the measurements. The maximum difference is 18 per cent for the ruby glass, a very saturated red. In the cases of the other four filters the differences do not exceed 8.2 per cent, a very satisfactory agreement. It does appear, however, that the differences are systematic, as was pointed out above. If it was assumed that the normal value of Y/B was 1.07 instead of 1.0, all differences between observed and computed values would be much smaller, or within 1 per cent for the light red, amber and blue filters, the average difference for all five filters being 2.4 per cent. However, there appears to be no valid reason why this should be done. The observed values are most nearly in agreement with the results obtained by computation when the visibility data of Hyde, Forsythe and Cady are used.

The values of J_λ and T_λ entering into the computations are believed to be substantially correct. The visibility factor for the two ends of the spectrum is subject to considerable uncertainty, and may account for a part of the differences found, although not all.

In practical photometry we are usually concerned with color differences much less than those represented by these five color filters. The most common color difference is that between incandescent lamps operated at different efficiencies. Some years ago the author participated in some transmission-factor determinations for a series of blue glasses used to filter the light from carbon lamps to match the light from tungsten lamps at various efficiencies. One glass represented the color difference between carbon and high-efficiency gas-filled tungsten lamps. For this glass eight observers obtained a transmission-factor of 0.227 with the contrast photometer, and a "normal" value of 0.214 with the flicker photometer, a difference of 6.3 per cent.

Since the flicker photometer has been shown (See Table 3) to overrate the blue, it appears probable from the evidence cited that the contrast photometer overrates blue by a still greater amount, if values

[8] Gibson, J.O.S.A., p. 113; Aug. 1924.

computed from the adopted visibility data are assumed to be correct. Direct experimental evidence as to the amount and type of disagreement between the contrast photometer and computed data similar to that given in this paper for the flicker photometer would be of considerable interest and value.

SUMMARY AND CONCLUSIONS

The transmission-factors of red, amber, green and blue glasses have been measured by a flicker photometer, using six observers of varied color-vision characteristics. From their observations have been determined the transmission-factor values for an observer of "normal" or "average" color vision.

Transmission-factor values for the same glasses have been calculated from spectrophotometric observations.

The color differences involved in the measurements of transmission-factors of these color filters are much greater than those usually encountered in routine photometric tests. With a "direct-comparison" photometer they would be practically unmeasurable by any one other than an experienced observer and even then the experimental error would undoubtedly exceed the differences shown in Table 3. With the flicker photometer six observers, four of whom had never before used a flicker photometer, were able to make observations fairly consistent with each other, as shown in Figs. 1–3.

In these tests the flicker photometer was used at a practical speed, namely, the minimum speed at which definite judgments of zero flicker or minimum flicker could be made. This speed depended upon the color and saturation of the color filter being measured. Tests of the effect of a speed 50 per cent greater, when measuring one of the flicker photometer color solutions, gave a difference within the range of experimental error.

Although the flicker photometer has been shown to give results which are systematically different from those obtained by computation from spectrophotometric data, these* differences are below 8.5 per cent for four out of five color filters, all of which present large color differences. This amount of disagreement does not appear to be excessive, in view of the difficulties of measurement, and is probably no greater than the experimental errors by any other method. It seems quite probable that the disagreement would be as great or greater for the contrast type of

* Since the preparation of this paper three more glasses have been studied, and the average difference between the computed and observed values for all eight glasses was only 5.8 per cent.

photometer. Hence, the author believes that no serious error would ever result from accepting unqualifiedly the results of measurements by the flicker photometer, used in the manner described above. Certainly no better alternative is at present available.

LIGHTING RESEARCH LABORATORY,
NATIONAL LAMP WORKS OF GENERAL ELECTRIC CO.,
NELA PARK, CLEVELAND.

Investigations on Low-Voltage Arcs.—The experiments were conducted with a bulb containing a circular Ni anode and a filament cathode; the observations were made upon current vs. voltage curves and upon the emission spectra of the discharge; corrections to the observed voltage were made to allow for contact P.D. and for initial velocities of the thermionic electrons; an oscillograph was used to detect and measure oscillations. The (i, V) curves exhibited discontinuities which are designated "striking" or "breaking" potentials according as the voltage was being raised or lowered when they occurred. In pure H_2 or pure N_2 there is only one breaking potential, which has a minimum value for a certain gas pressure (around 0.5 mm for H_2) which minimum is identified with the ionizing potential of the gas. For H_2 it is 16.2 volts, in good agreement with other estimates; for N_2 it is apparently just the same, but actually there are oscillations whenever the apparent voltage lies between 16.2 and 16.9, so that 16.9 volts is supposed to be the true ionizing potential for N_2. In hydrogen the observed (i,V) curve departs from a 3/2 power curve rather abruptly at 13.6 to 13.8 volts, which is taken to show incipient ionization of atomic hydrogen around the filament. In mixtures of H_2 and Hg vapor the (i,V) curve sometimes showed two discontinuities instead of one as V was being raised, and (more rarely) two discontinuities instead of one as V was being lowered. These did not lie at the same voltages as for the two gases individually, although it happens that the voltage difference between the two discontinuities in the curve for rising V is equal to the ionizing potential of Hg. The article contains accounts of a great amount of observations, not always clearly described and not easy to interpret; notably a quantity of observations on the relative intensities, under different conditions, of various bands commonly attributed to ammonia, leading *inter alia* to the conclusion that the "Schuster bands" in the yellow are due to a complex of ammonia and water molecules.—[C. T. Kwei, (Princeton); Phys. Rev., (2), *26*, pp. 537–560; 1925.]

KARL K. DARROW

Low-Voltage Arcs in Hg, He, and Ne.—These are arcs which continue in existence when the voltage between anode and hot-filament cathode is lowered below the resonance potential of the gas; the possibility of oscillations bringing the voltage momentarily above the resonance potential being excluded, as Bär thinks, by alternative measurements with a D.C. and an A.C. instrument and by the failure of the shunting of a $2\mu f$ capacity across the tube to make any difference. Bär experienced much difficulty through the potential drop across his filament and through its habit of burning out; but he was able to show that an arc could be maintained at 2.25 volts in Hg and at 4 volts in Ar (resonance potentials 4.7 and 11.6 respectively), although this result with Ar could not be attained without the highest possible temperature of the wire and purity of the gas and at a high pressure (10 mm); it was necessary to form the arc between gold foil electrodes in a tube submerged in liquid air. With helium it was twice and only twice possible to go down below the resonance potential (to 16.5 volts), by using the same precautions as to hotness of wire and purity of gas Holst and Oosterhuis brought the arc-maintaining voltage in Ar down to 3.5 and in Ne down to 7.5. Bär remarks that the oscillatory arc in He cannot be sustained unless the tube is in liquid air, which is attributed to the excited He atoms losing their energy in collisions of the second kind with molecules of residual vapors before the current-cycle is completed.—[R. Bär, Zs. f. Phys., *31*, pp. 430–438; 1925.]

KARL K. DARROW

Negatively-Resulting Attempt to Determine the Electric Moment of Molecules of Liquocrystalline Substances.—It is well-known that magnetic fields orient the long narrow molecules (or molecule-aggregates) of certain liquid crystals with their long axes parallel to the field, the parallelism being practically perfect at quite attainable field strengths. The molecules are therefore magnetic doublets; are they electric doublets likewise, with the magnetic and the electric poles at their extremities? If so, a substance of this class held between metal plates should become electrically polarized when a magnetic field is applied normal to the plates, and an electrometer connected between these should reveal a P.D. No such effect was found with any of five common liquocrystalline substances. The calculations are not so given that one can immediately tell how low is the minimum polarization below which the actual value then must lie; but it is less than 10^{-6} as great as the value predicted from theory by Born.—[G. Szivessy, Münster; Zs. f. Phys., *34*, pp. 474–484; 1925.]

KARL K. DARROW

INSTRUMENT SECTION

HELMHOLTZ COILS FOR PRODUCING UNIFORM MAGNETIC FIELDS*

By Arthur Edward Ruark, Associate Physicist,
and
Melville F. Peters, Associate Physicist

ABSTRACT

The "Helmholtz coil" is really a pair of coils so proportioned as to produce a magnetic field uniform in magnitude and direction over a large region. The theory of the magnetic field produced by such a coil is briefly developed and the final (series) formula is put into a form convenient for calculations. For ordinary applications only the field at the center is required. A convenient design, developed for obtaining soft x-ray potentials by determining the velocities of ejected photo-electrons, is described, and is further explained by photographs and drawings.

I. INTRODUCTION

By the use of a solenoid, wound on an ellipsoidal form, it is possible to produce a magnetic field which is uniform in magnitude and direction, to a high degree of approximation, throughout a considerable volume; but such a solenoid is inconvenient when it is necessary to have frequent access to apparatus placed in the uniform field. This is the case in studies of the polarization of resonance radiation, or in researches requiring the magnetic resolution of streams of ions or electrons. Under such conditions, the double coil developed by Gaugain and by Helmholtz, which bears the name of the latter, is to be preferred because of its open construction. The literature concerning these coils is rather scattered,[1] and the information required for their design does not seem to have been assembled in a form suitable for facile computations.

In "Practical Physics," Professor W. Watson has given a formula for the field in the median plane of a Helmholtz coil, but Gray[2] gives a

* Published by permission of the Director of the Bureau of Standards of the U. S. Department of Commerce.

[1] Reference to the earlier literature will be found in Wiedemann's "Die Lehre von der Elektrizitat," Vol. III, p. 275. A sine galvanometer of extremely careful construction in which these coils are used has been described by S. J. Barnett, Researches of the Dept. of Terrestrial Magnetism, *4* (Publication No. 175 of the Carnegie Institution of Washington, 1921).

[2] Absolute Measurements in Electricity and Magnetism, 2nd edition, pp. 210–20, Eq. (23), 1921.

different result. We find from Gray's Eq. (22) that the second term in Eq. (23) contains a typographical error. The coefficient $2^5/5^3$ should be $2^2/5^3$. Starting from the fundamental equations and proceeding by an independent method, we have obtained an equation which agrees with that of Gray, thus amended, and conclude that Watson's formula is incorrect.

Gray gives the field at any point in space due to a single circular turn, but his formula for the Helmholtz coil applies only to the field in

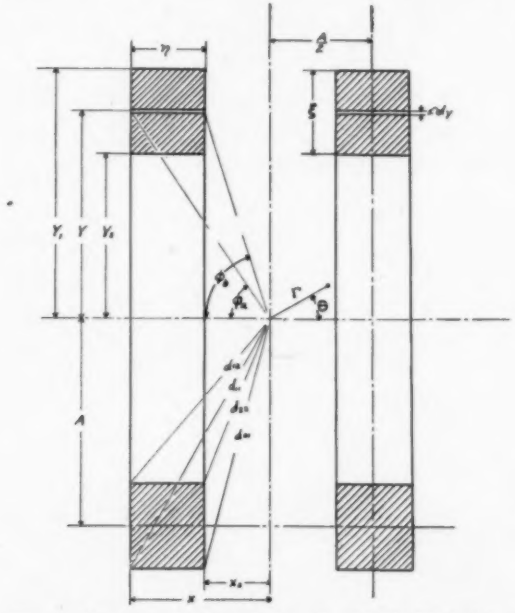

FIG. 1. *Diagrammatic Cross Section of an Idealized Helmholtz Coil.*

the median plane of the coil, and therefore gives no intimation as to the homogeneity of the field at points outside the median plane.

Ellis, Clarkson, and Rainy[3] made an experimental study of the field of a Helmholtz coil but did not compare their results with any formula.

In this paper we shall collect all formulas needed to make the design of a Helmholtz coil a matter of routine, and shall describe a coil of convenient proportions constructed in this laboratory.

[3] Proc. Roy. Soc. of Edinburgh, *13*, p. 523, 1885.

II. THE FIELD OF A HELMHOLTZ COIL

We shall begin by writing down the scalar magnetic potential, ω, due to an idealized coil having a uniform distribution of current density over the cross sections of its windings, as given by Maxwell.[4] After the study of such an idealized coil, we shall discuss the errors introduced by the discrete distribution of the wire.

In Fig. 1, let A be the distance between the mean planes of the coils, which is also equal to their mean radius. The origin of coordinates is taken at the middle point of their common axis. The typical point at which the magnetic field H is to be evaluated has coordinates r and θ. The components of H along the directions of increasing r and increasing θ are

$$H_r = -\frac{i}{10}\frac{\partial\omega}{\partial r}, \qquad H_\theta = -\frac{i}{10}\frac{1}{r}\frac{\partial\omega}{\partial\theta}. \tag{1}$$

Here the factor 10 is introduced because the current i is expressed in amperes. We shall obtain the field only at points for which r is less than A, so ω and H may be expanded in ascending powers of r/A. Each coil is of rectangular cross section, with breadth η and thickness ξ. Let n denote the number of windings in *both coils together*. Then, to a sufficient approximation,

$$\begin{aligned} -\omega = \text{const} &+ \frac{16\pi n}{5\sqrt{5}A}\left(1 - \frac{1}{60}\frac{\xi^2}{A^2}\right) rP_1(\theta) \\ &+ \frac{0.0512\pi n}{3\sqrt{5}A^3}(31\xi^2 - 36\eta^2)\frac{r^3}{A^2}P_3(\theta) - \frac{0.73728\pi n}{\sqrt{5}A}\frac{r^5}{A^4}P_5(\theta) \end{aligned} \tag{2}$$

In this equation the constants 0.0512 and 0.73728, as given by Maxwell, have the significance $2^5/5^4$ and $2^8 3^2/5^5$ respectively. The P's are Legendre's coëfficients, defined by

$$P_1(\theta) \equiv \cos\theta\,, \quad P_3(\theta) \equiv \frac{\cos\theta}{2}(5\cos^2\theta - 3)\,, \quad P_5(\theta) \equiv \frac{\cos\theta}{8}(63\cos^4\theta - 70\cos^2\theta + 15)$$

B. (1)

$$H_r = \frac{i}{10}\cos\theta\left[\frac{16\pi n}{5\sqrt{5}A}\left(1 - \frac{\xi^2}{60A^2}\right) + \frac{0.0512\pi n}{2\sqrt{5}A}\cdot\frac{31\xi^2 - 36\eta^2}{A^2}\frac{r^2}{A^2}(5\cos^2\theta - 3) - \frac{0.73728\sqrt{5}\pi n}{8A}\frac{r^4}{A^4}(63\cos^4\theta - 70\cos^2\theta + 15)\right]$$

[4] Electricity and Magnetism, 3rd edition, *2*, pp. 333, 337, and 356.

$$H_\theta = \frac{i}{10}\sin\theta\left[-\frac{16\pi n}{5\sqrt{5}A}\left(1-\frac{\xi^2}{60A^2}\right)-\frac{0.0512\pi n}{2\sqrt{5}A}\cdot\frac{31\xi^2-36\eta^2}{A^2}\cdot\frac{r^2}{A^2}(5\cos^2\theta-1)\right.$$
$$\left.+\frac{0.73728\sqrt{5}\pi n}{8A}\cdot\frac{r^4}{A^4}(63\cos^4\theta-42\cos^2\theta+3)\right]$$

Now $$H_x = H_r\cos\theta - H_\theta\sin\theta\,, \qquad H_y = H_r\sin\theta + H_\theta\cos\theta\,. \tag{3}$$

$$\therefore H_x = \frac{ni}{10A}\left[\frac{16\pi}{5\sqrt{5}}\left(1-\frac{\xi^2}{60A^2}\right)+\frac{0.0512\pi}{2\sqrt{5}}\cdot\frac{31\xi^2-36\eta^2}{A^2}\cdot\frac{r^2}{A^2}(3\cos^2\theta-1)\right.$$
$$\left.-\frac{0.73728\sqrt{5}\pi}{8}\,\frac{r^4}{A^4}(35\cos^4\theta-30\cos^2\theta+3)\right]$$

$$= \frac{ni}{A}\left[0.44959\left(1-\frac{\xi^2}{60A^2}\right)+0.0035967\frac{31\xi^2-36\eta^2}{A^2}\,\frac{r^2}{A^2}(3\cos^2\theta-1)\right.$$
$$\left.-0.06474\frac{r^4}{A^4}(35\cos^4\theta-30\cos^2\theta+3)\right] \tag{4}$$

$$H_y = \frac{ni}{10A}\sin\theta\cos\theta\left[-\frac{0.0512\pi}{2\sqrt{5}}\cdot\frac{31\xi^2-36\eta^2}{A^2}\,\frac{r^2}{A^2}\right.$$
$$\left.+\frac{0.73728\sqrt{5}\pi}{8}\,\frac{r^4}{A^4}(28\cos^2\theta-12)\right]$$

$$= \frac{ni}{10A}\sin\theta\cos\theta\left[-0.0035967\frac{31\xi^2-36\eta^2}{A^2}\,\frac{r^2}{A^2}\right.$$
$$\left.+0.06474\frac{r^4}{A^4}(28\cos^2\theta-12)\right] \tag{5}$$

It is customary so to choose the dimensions of the cross section that $31\xi^2-36\eta^2$ is very small, thereby practically eliminating the terms containing r^2/A^2.

The field at the center of the coil may be checked by using a closed formula, obtainable by simple integration. This formula is given in Gray's Absolute Measurements, 2nd edition, p. 216. Let there be t turns of wire per centimeter, so that each square centimeter carries the current t^2i. The current is supposed to be uniformly distributed over the cross section of the winding. All other symbols used can easily be identified in Fig. 1. Each elementary layer of thickness dy will

carry a current $l^2 i\, dy$ per unit of length. The formula for the field at 0 due to such a layer, is

$$dH = \frac{2\pi l^2 i dy}{10}(\cos\varphi_a - \cos\varphi_b)$$

Integrating from y_2 to y_1, and doubling to include the contributions of both coils, we have

$$H = \frac{4\pi l^2 i \log_e 10}{10}\left[x_1 \log_{10}\left(\frac{y_1+d_{11}}{y_2+d_{12}}\right) - x_2 \log_{10}\left(\frac{y_1+d_{21}}{y_2+d_{22}}\right)\right],$$

where i is in amperes.

The question now arises as to what modifications are introduced by the finite size of the wire. Calculations similar to those of Rosa[5] show at once that formula (4) may certainly be used with an accuracy exceeding 1 part in 10,000 in all practical cases, provided only that all layers of the windings are similar. Usually, however, the coils are wound in the manner shown in Fig. 2, with each layer running in the

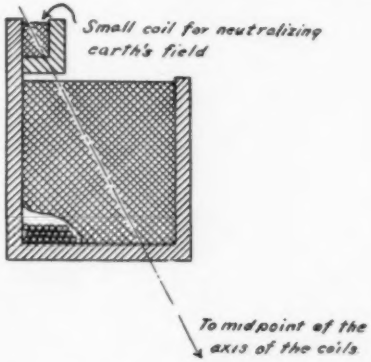

FIG. 2. *Cross Section of Winding of a Helmholtz Coil Suitable for Obtaining Magnetic Spectra of photoelectrons ejected from solids by soft x-rays.*

grooves formed by the layer beneath it. To obtain upper limits for t[illegible] thus introduced, we first evaluate the correction terms of [illegible] using for ξ and η the actual dimensions of the winding. We do the [illegible]me for a coil having one more layer but one less turn per layer, and for a coil having one less layer but one more turn per layer. In this way we find that for the coil described in Section III, the error does not exceed 3 parts in 10,000 at points for which $r = A/2$. This degree of accuracy is more than sufficient for our purpose.

Rosa[5] has shown that it is best to use for ξ and η the overall dimensions of the winding; that is, the insulation of the outside turns is to be included.

We shall now discuss the changes which must be made in our formulae if we use coils which deviate slightly from the condition that the distance, $2B$, between median planes is equal to the radius. Let $C^2=A^2+B^2$. Then formulae (4) and (5) must be replaced by the following:

$$H_z=\frac{ni}{10}\left[G_1+\frac{3}{2}G_3r^2(3\cos^2\theta-1)-\frac{0.73728\sqrt{5}\pi}{8}\frac{r^4}{A^5}(35\cos^4\theta-30\cos^2\theta+3)\right] \tag{4'}$$

$$H_\rho=\frac{ni}{10}\left[-\frac{3}{2}G_3r^2\sin\theta\cos\theta+\frac{0.73728\sqrt{5}\pi}{8}\frac{r^4}{A^5}\sin\theta\cos\theta(28\cos^2\theta-12)\right] \tag{5'}$$

where

$$G_1\equiv 2\pi\frac{A^2}{C^3}\left\{1+\frac{\xi^2}{24}\left(\frac{2}{A^2}-\frac{15B^2}{C^4}\right)+\frac{\eta^2}{8}\frac{4B^2-A^2}{C^4}\right\}$$

$$G_3\equiv\frac{4\pi}{C^7}A^2\left(B^2-\frac{A^2}{4}\right)+\frac{\pi}{24}\frac{\xi^2}{C^{11}}\{C^4(8B^2-12A^2)+35A^2B^2(5A^2-4B^2)\}+\frac{5\pi\eta^2A^2}{8C^{11}}A^2\{A^4-12A^2B^2+8B^4\}$$

The last term of (4) and (5) has been left unaltered since small errors in a small correction term may be neglected.

The design of the coil now depends on three factors: (1) the degree of homogeneity required over the region occupied by the apparatus to be placed within the coil; (2) the dimensions of that region; (3) the field strength required.

For example, suppose the coil is to be used to produce a field in a tube 12 cm long for the study of resonance radiation, and that this field must be uniform to 1 part in 1000. The reader will easily convince

[5] Bulletin of the Bureau of Standards, *2*, p. 71, 1906, and *3*, p. 209, 1907.

himself that when uniformity of magnitude is properly taken care of, uniformity of direction will exceed all requirements; that is, H_y will be many times smaller than is really necessary. For a properly proportioned coil, we may generally neglect the term in r^2/A^2. Then the ratio of the correction terms to the first term in the expression for H_x is about

$$0.144 \frac{r^4}{A^4} (35\cos^4\theta - 30\cos^2\theta + 3)$$

Putting $r = 6$ cm, and taking the value of θ appropriate to the position of an end of the resonance tube, we find the value of A which will make this expression equal to .001. The current which will produce the required field at the center of the coil is then determined by the approximate formula

$$H = 0.45ni/A .$$

The cross section of such a coil may well be chosen to give a current density of 1000 ampere turns per sq. in.; that is, about 160 ampere turns per sq. cm. is a conservative value which will prevent undue heating. When the cross section and the voltage at which the coil is to be operated are known, the proper size of wire may be selected by methods given in any electrical handbook.

III. DESCRIPTION OF A CONVENIENT HELMHOLTZ COIL

We have constructed a Helmholtz coil, with a radius of about one foot, for obtaining the magnetic spectra of the relatively slow photoelectrons ejected from solids by soft x-rays. The field at the center of the coil is about 200 gauss when the potential applied to each of the two windings is 77 volts. This corresponds to a field of 28.8 gauss per ampere. Each winding has 816 turns of No. 14 enamel and cotton covered magnet wire, laid in 32 layers of 26 and 25 turns alternately. Fig. 2 shows the cross section of one of the main windings and of a smaller winding mounted concentrically and used to neutralize the earth's field. This eliminates all computations to correct for the effect of the earth's field, and makes it very easy to check up the neutralization of this field by means of a flip coil. Provision is made for mounting an electron spectrograph at the center of the coils. Fig. 3 shows the completed coil, so mounted that it can easily be oriented with its axis parallel to the earth's magnetic field.

All parts of the coil have been machined from brass or phosphor bronze castings, made in the Bureau of Standards' foundry. Samples

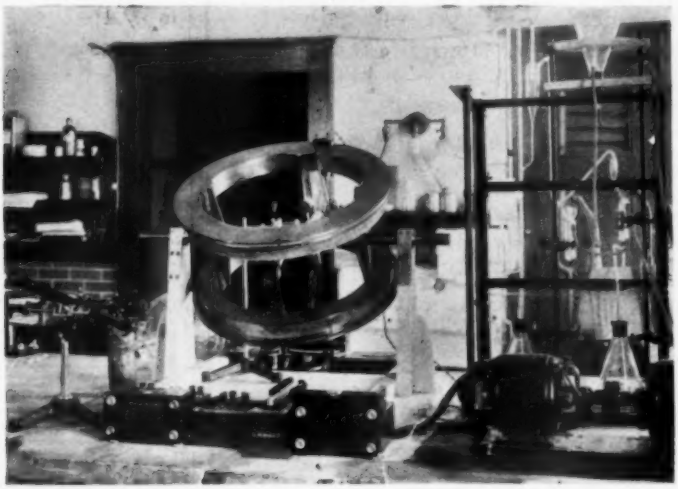

FIG. 3. *Helmholtz coil constructed at the Bureau of Standards for obtaining magnetic spectra of photoelectrons.*

of the materials used were tested by Dr. R. L. Sanford of the magnetic section of this Bureau; it was found that their susceptibilities are less than $5 \cdot 10^{-5}$, a satisfactory state of affairs.

BUREAU OF STANDARDS,
WASHINGTON, D. C.,
FEBRUARY 10, 1926.

FREE FALL APPARATUS

By Leo Behr and F. W. Reynolds

ABSTRACT

A description is given of apparatus constructed for undergraduate use in studying the laws of uniformly accelerated motion and in determining the value of gravity. The falling body has no constraints and the time may be measured with an ordinary watch.

In the undergraduate laboratory work here, the laws of uniformly accelerated motion and incidentally the value of gravity, have been directly determined with the usual apparatus consisting of a falling tuning fork which in its descent traces a wavy line on a coated sheet of plate glass. The experimental results have not been entirely satisfactory, not only because the values for g were much too low but also because the plot of speed against time departed very materially from a straight line. Pedagogically the experiment also labored under the disadvantage that the student did not determine for himself the frequency of the fork. At the request therefore of Professor Richtmyer, the present apparatus was built in which the falling body is completely free and the timing interval can be measured directly with a speed counter and a watch.

The apparatus is shown in the photograph and in Fig. 1. To the machined face of an I-beam is clamped a sheet of paper, and in front of these and insulated from the beam is mounted a brass strip. By means of an arrangement presently to be described, a high voltage is applied between the strip and the beam for a very short time and at constant intervals of approximately one thirtieth of a second. The weight is supported by an electromagnet and upon being released, falls close to both the beam and the strip, allowing a spark to jump from the latter to the weight and from there through the paper to the beam. The projecting rim definitely locates the spark with respect to the weight so that the holes in the paper form a record of the positions of the body at the times of sparking.

Fig. 1 is drawn to scale and the dimensions may be estimated from the fact that the overall length of the weight is six inches. Its shape was dictated largely by the convenience of the mechanic who constructed it. The plane of the projecting rim preferably contains the center of gravity of the falling body. The only essential feature is that the topmost portion be of steel with its end smoothly rounded off approxi-

mately hemispherical. If the rest of the weight is of steel also, there must be inserted a section of brass as otherwise the attraction for the beam is sufficient to cause an appreciable deflection. The supports of the magnet itself are also of brass. The core has pressed in at its lower end a quarter inch steel ball which forms nearly enough a point support for the weight so that when released the latter drops without any appreciable angular motion.

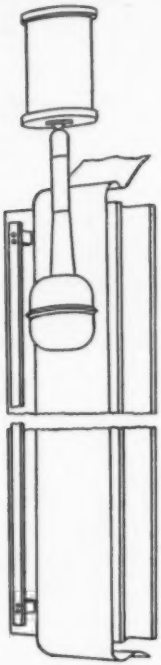

FIG. 1

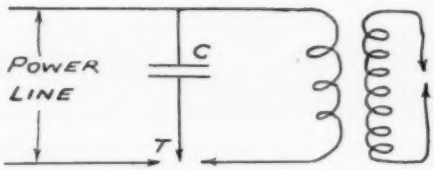

FIG. 2

The records were first made on ordinary adding machine paper but this was unsatisfactory as the holes punctured in the paper were rather small and easily confused with the numerous imperfections of the paper itself. Smoking the paper eliminated this difficulty as wherever the spark jumped through, it removed the soot around the hole over an area sufficiently large to make detection easy. The only objection to this procedure is the nuisance of having a large number of undergraduates making and handling the smoked paper. Since then a thin blue paper has been found which has a translucent coating of paraffin.

The passage of the spark melts the paraffin, leaving a blue spot in which the spark hole itself is easily found.

The diagram of connections for producing the timing spark is shown in Fig. 2. By means of a motor driven commutator T, the bank of condensers C, is, alternately connected to the power line and discharged through the primary of an automobile spark coil. One end of the secondary is connected to the beam and the other end goes to the insulated strip. For driving the commutator at a constant speed it has been found convenient to use a small self-starting self-excited 1800 rpm synchronous motor. The condensers are the paper ones as employed in telephone service and are rated at 500 volts. Ten microfarads are ample. To ensure certain sparking at every revolution it was found necessary, with the spark coil used, to charge the condensers to 250 volts. Although it is probably not difficult to construct a coil which would operate at a lower voltage, it was easier in our case to merely secure the required potential. A small transformer steps the 110 volt ac up to 180 volts and the contact on the commutator is set to charge the condensers to the peak voltage.

No special precautions have been taken for guarding the student against the high voltage as the penalty for carelessness is not a serious one When the apparatus was first installed it was noticed that occasionally the men would stoop to smooth out a wrinkle in the paper after the weight had been hung up. In this position a failure of the magnet would drop the weight on the student's head. The portion of the beam below the magnet is therefore now enclosed in a cage with a switch so connected to the door as to make necessary the shutting of the door before the magnet circuit can be closed.

In a casual inspection of about one hundred students' curves of speed against time, it was seldom that a plotted point was found which departed from a straight line by more than one per cent of the maximum speed. The results obtained for g were usually well within one per cent of the accepted value and with possibly one or two exceptions were all within two per cent. A sufficient cause for this variation is the inconstancy of the local power supply frequency as the fall of the weight occurs in about half a second while the determination of the motor speed, using an ordinary watch and speed counter, requires at least three minutes. No attempt has been made to reduce the initial measured velocity to zero by synchronizing the release of the weight with the occurrence of the spark, as it was considered a positive advantage

for the speed-time curve to have an intercept for the student to properly interpret.

CORNELL UNIVERSITY,
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Light and Health, a discussion of light and other radiations in relation to life and to health. By M. Luckiesh and A. J. Pacini, pp. vi+302. $5.00. Baltimore, The Williams and Wilkins Company. 1926.

This book is avowedly a pioneering reconnaissance of a field in which there is much scattered information but no thorough and systematic treatment now available. It is intended to be a plain story, simply told, of light and its effects on the health and happiness of mankind, "suited for general readers, mothers, nurses, public health officials and others interested in health as well as for many scientific men not specialized in the subject, including physicians, hygienists, biologists, and physiologists."

Having aimed at such diverse classes of readers the authors have evidently felt at liberty to include a wide variety of material. The scientific student seeking established facts may find it somewhat burdensome to sort them out from the speculative setting in which they are placed; on the other hand the general reader will probably have some difficulty in following the complex interrelation set forth as connecting ultraviolet radiation, the sympathetic nervous system, calcium metabolism and the glandular secretions, which in turn are supposed to control the type of personality as well as the health of the individual. The fact that there is so much interrelation between different organs and functions of the body makes it difficult to treat the effects of radiation on any one of them separately, as is done in a series of chapters in this volume. This plan is perhaps as good as could be suggested, but the reader is likely to be confused by lack of coordination between the several chapters, which goes so far in some cases as to involve practical repetition of material. One gets the impression that the text has been hastily put together, and this impression is reinforced by the rather poor editing and proof reading which have left many errors of grammar and typography. One expects from the Waverly Press better work in these respects. In choice of paper, type and general make-up the volume is, however, highly creditable, and these details as well as the easy, flowing style in which most of the book is written help to make it thoroughly readable.

In brief, one may wish that the authors had given somewhat more time and labor so as to make the book a more perfect literary work or a more complete and usable reference volume; yet as it is they have produced a book which can not be ignored by those who aspire to be well informed regarding either the applications of light or modern practice in medicine and hygiene.

E. C. CRITTENDEN

DIRECT READING IONIZATION GAUGE

By Clifton G. Found and Neil B. Reynolds

In a previous paper[1] Dr. Dushman and one of the authors described the operation and calibration of a three electrode tube as an ionization gauge for pressure measurements. Its operation is dependent on the fact that the positive ionization is proportional to the pressure of the gas, in the tube, when the electron current and accelerating voltage are maintained constant. This gauge is shown in Fig. 1.

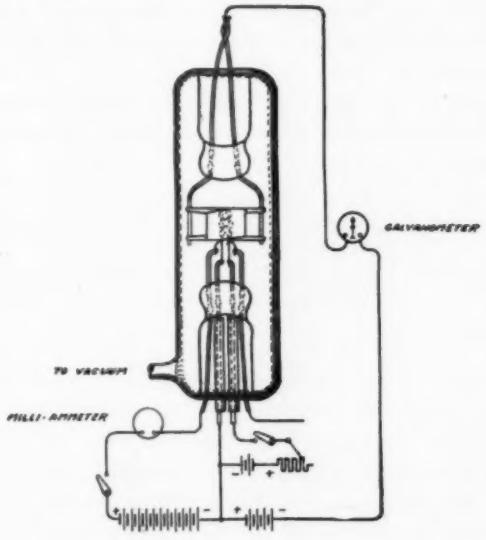

Fig. 1.

Later it was thought desirable to have a gauge simpler in construction and easier to manufacture. The greatest simplification in manufacture is obtained by mounting all the electrodes on a single stem similar to the construction of the small radio receiving tubes. Accordingly, various types of radio receiving tubes were tested as ionization gauges.

[1] Dushman and Found, Phys. Rev., *17*, p. 7, 1921; also Found and Dushman, Phys. Rev., *23*, p. 734; 1924.

The calibrations were made with air by comparing the tubes with a gauge of the type shown in Fig. 1. The two tubes were attached by a fork to the same exhaust system and simultaneous observations taken. Previous to the calibration, both gauges were given a one-hour oven bake-out at 360°C. The metal parts were thoroughly degassed. For the standard gauge, the process was that described in the previous paper[2] and for the other type, the metal parts were heated to a high temperature by means of a high frequency furnace.

Electrical Connections. In using the new tube as a gauge, the general practice is to use the grid as anode or accelerating electrode with an accelerating potential of 125 volts. The plate is used as collector at a voltage of about 22 volts negative with respect to the negative end of the filament. The electron current is controlled by variation of the cathode temperature.

Calibrations. Table 1 gives the results for calibrations of a number of standard radio tubes and the conditions under which these calibrations were made.

TABLE 1

Tube	Accelerating Volts	Electron current in milliamps	Ionization in microamps/bar
Standard			
"Ionization guage"	125	0.5	1.4
	250	20.	46.
UV-201	125	0.5	2.0
UV-204-A	125	0.5	2.3
	250	10.	47.
UV-206	125	0.5	1.2
	250	10.	35.
UV-208	125	0.5	1.8
	250	10.	37.
UV-200	125	0.5	1.7
	125	10.	34.

DIRECT READING INSTRUMENT

INTRODUCTION

From experience with the use of the ionization gauge for general pressure measurements, such as on exhaust systems, there has been shown a need for a portable instrument which can be readily attached

[2] Dushman and Found, Phys. Rev., *17*, p. 7, 1921; also Found and Dushman, Phys. Rev., *23*, p. 734, 1924.

to the gauge and which will read the pressure directly without further adjustments. Such an instrument should be capable of steady operation over a period of time in order to follow the progress of exhaust. It is also desirable that a more robust instrument than a galvanometer be used to indicate pressures. Moreover, it would simplify the operation greatly if such an instrument could be operated directly off a 125 volt line and not be sensitive to small fluctuations in the line voltage. We have attempted to incorporate these features in the instrument described in this paper.

The electrical connections of the instrument are shown in Fig. 2. As ionization gauge, a UV-200 tube was chosen. This is represented

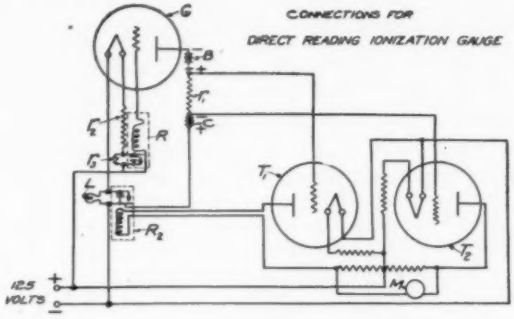

FIG. 2.

in the diagram as G. The filament requires about 1 ampere heating current. This is supplied from 125 volt line through a resistance r_2 in the positive arm. The accelerating voltage to the grid is supplied from the same source, the positive terminal being connected to the grid through the coil of a relay or Kearsley stabilizer,[3] R_1, across the contacts. The relay is adjusted so that when the electron current to the grid increases above 10 m.a. the contacts open, throwing the resistance r_3 in series with the filament, thus decreasing its temperature and maintaining the electron current in the gauge constant at 10 m.a. The plate is used as collector and is held 27 volts negative by means of the batteries B and C.

In order to amplify the positive ionization currents, a resistance r_1 is placed in series with the collector and the voltage drop across this is amplified by means of a push-pull amplifier circuit using UV-201-A radiotrons.

[3] The Journal of Radiology, July, 1921.

DESCRIPTION OF AMPLIFIER

The push-pull amplifier circuit is well known, but a brief description may not be out of place here.

It consists of two similar tubes with series resistances in the plate circuits. These resistances are adjusted so that no current flows in a milliammeter placed between the plates of the two tubes. However, if the potential of the grid of one is raised, the balance is destroyed and a

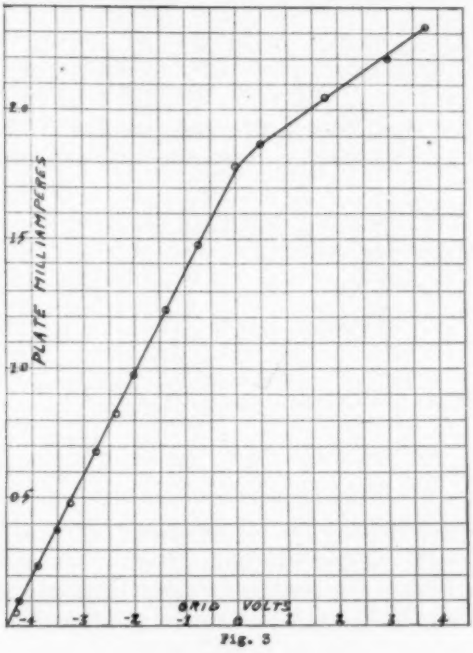

FIG. 3.

current is registered in the milliammeter. This circuit has the advantage that the output current is linearly related to the change in grid voltage, as long as the grid is not made positive. When this occurs, the output current increases more slowly until nearly a constant value is reached.

In the present case, the grids are normally maintained at a negative bias of 4.5 volts. The grids of the tubes, however, are connected to opposite ends of the resistance r_1 (125000 ohms), so that when positive ion current flows to the collector of the gauge the voltage of the grid

of T_1 is made more positive. This increases the plate current in this tube, destroying the balance of the system, and causes a current to flow through the milliammeter (M). This current is proportional to the change in grid voltage, as shown in Fig. 3 and consequently to the positive ionization current, which, in turn, is proportional to the pressure.

PROTECTION FOR THE GAUGE

In order to prevent possible burn out of the gauge filament, should it be lighted when too much gas is present, a second relay R_2 is inserted in the plate circuit of the amplifier T_1. This is so adjusted that when this current reaches a value equivalent to 10 bars pressure, the contacts of the relay open the gauge filament circuit and at the same time light a red lamp as a warning signal.

The complete instrument may be operated on a.c. or d.c. It is quite insensitive to variations of plate voltage, the average error introduced being 0.5 per cent per volt.

Calibrations on both a.c. and d.c. were made and are given below:

Calibration

Let i_p = positive ionization current per bar
r_i = resistance in series with grid of T_1
E_g = change in grid voltage of T_1
Then $E_g = i_p r_i$

Let constant of amplifier be C_m.

$$C_m = \frac{I_a}{E_g}$$

where I_a measures output current.

Then $I_a = C_m i_p r_i$ per bar.

D.C. Calibration

Using ionization gauge at 10 m.a. electron current and 120 volts d.c. the following values were obtained:

$i_p = 34 \times 10^{-6}$ amps per bar
$r_i = 125{,}000$ ohms
$C_m = 0.32$ milliamperes per volt

Therefore

$I_a = 0.32 \times 125{,}000 \times 34 \times 10^{-6}$
$= 1.4$ milliamperes per bar

A.C. Calibration

For alternating current source of volts, the following constants were obtained:

$$i_p = 24\times10^{-6} \text{ amps per bar}$$
$$r_i = 125{,}000 \text{ ohms}$$
$$C_m = 0.36 \text{ milliamperes per volt}$$

Hence

$$I_a = 0.36\times125{,}000\times24\times10^{-6}$$
$$= 1.1 \text{ milliamperes per bar.}$$

RESEARCH LABORATORY,
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The Earth and the Stars. By C. G. Abbot, D.Sc., Director of the Smithsonian Astrophysical Observatory. xi+264 pages (including 32 plates and 46 figures). D. Van Nostrand Company, New York. $3.00 net.

This volume is intended, according to the preface, for "those who wish to acquire by easy reading a general survey of the universe they dwell in. . . .¶ From the structure of the atom to the structure of the universe, we seem to follow a definite thread of reason." It is the author's purpose to present, in simple language, the salient facts of this general scheme.

After three introductory chapters, describing the history of astronomy and the structure of the atom, comes a group of five chapters, over one third of the book, devoted to the earth, sun, moon, and the solar system. Then follows a very interesting chapter on "The Calendar, Star Places, Navigation." The remainder of the book is devoted to the stellar universe.

It requires but a glance through the book for one to realize how well the author has succeeded in presenting this very comprehensive subject, in a nontechnical, yet complete and concise manner. The book is a valuable addition to the growing number of books which aim to present the essential facts of science to the lay reader.

F. K. RICHTMYER

SOME DEFECTS OF CLINICAL THERMOMETERS

By Bradford Noyes, Jr.

Since the work of Waidner and Fischer[1] little has been published on Clinical Thermometers, and it was felt that it might be profitable to continue and extend the work of these authors. The present study deals largely with the inconsistent clinical thermometer, one which for any one of several reasons fails to repeat its readings.

Clinical thermometers belong to the class known as maximum registering thermometers; that is, the mercury remains at the maximum position reached since the column was last "set" or "shaken down." This is accomplished by means of what is commonly called a contraction. When the bulb is heated the mercury squeezes through a small opening and rises in the stem, but when the bulb is cooled the mercury thread is broken by the contraction and remains in the stem. Hence, the top of the mercury column indicates the maximum temperature reached.

The construction of the contraction requires great skill on the part of the workman. It is accomplished as follows:

The piece of thermometer tubing is heated at the proper point and air pressure is applied to the inside bore or capillary. This causes the capillary to expand into a chamber as shown in Fig. 1. This figure

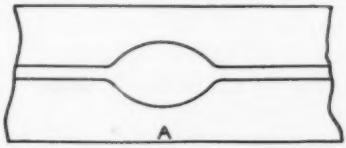

Fig. 1

greatly exaggerates the size of the bore, which may have a diameter of only .02 mm. After the bulb of the thermometer has been filled with mercury and the space above thoroughly evacuated, a fine flame is brought to play on point *A*, Fig. 1, which causes the wall of the chamber to collapse. The contraction then has the shape shown in Fig. 2. A cross section examined along line *B-B* has the general shape indicated by Fig. 3.

[1] "The Testing of Clinical Thermometers" Waidner and Fischer, Bulletin of Bureau of Standards, *1*, No. 2; 1904.

C and *D* represent small passageways around the collapsed section of wall. These passageways are not the same size, and often one is completely closed. It is this fine constriction which breaks the mercury thread and causes maximum registration.

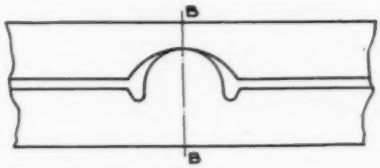

FIG. 2

The action of the contraction of a large number of thermometers was examined by means of a photo-microscope, an image of the contraction being projected on a screen. The bulb of the thermometer was heated slowly, and the action of the mercury as it passed through the contraction was studied. By far the greater number showed that if

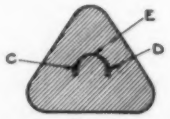

FIG. 3.

the bulbs were heated uniformly the mercury in the stem did not rise continuously but was increased by small definite increments, the size of which depended largely upon the individual contraction of the tube. This phenomenon may be explained if we consider the mercury as acted upon by two forces, gravity and surface tension. The globule squeezes through the most constricted point of the tube into a section where the opening becomes larger and more circular in cross section. That mercury, which has passed the most constricted portion, tends to take the position where its potential energy is at a minimum. As is well known, mercury has a very high surface tension, and the increase of potential energy due to increased height may be small compared to the decrease in surface energy which occurs when the mercury rises in the tube above the contraction. Hence, drops are formed similar to drops of water formed under the action of gravity, and, as would be expected, their size depends largely upon the shape of the individual contraction.

Tubes which exhibit this phenomena are known as "Spitters," a term which may be defined as "A tube in which the mercury in the bulb is never in contact with the mercury above the contraction, providing the tube is heated slowly." It is evident that the rise of the mercury in such tubes is discontinuous, and hence, such tubes cannot be expected to be entirely consistent. A large number of tubes rejected because they "did not repeat" were examined. Each was found to be a "spitter" in which the globules were quite large in size. The scale length equivalent to each globule or "spit" was found by noting the increased reading of the thermometer when a given number of globules passed the constriction. In some cases, a globule was found to represent as much as .35°F. Assuming these globules to be of equal size and that a tube were properly pointed and graduated, the maximum error of such a tube would be .35°F. The probable error in such a case is .28 of the maximum error, or .098°F. A number of such thermometers was studied and in no case was the observed difference greater than this maximum difference. Moreover, the computed probable error agreed as well as could be expected with the probable error obtained from the above theory. Hence, it would seem that most of the eccentricities of tubes may be blamed on contractions which allow large globules to pass. A "spitter" in which the size of the drops is small is to be desired for such a tube is not liable to retreat.

The effect of gravity on the mercury of most tubes proved to be extremely small. Occasionally, however, a tube was found in which this effect caused very inconsistent results.

The method of study was simple. A thermometer was mounted in a micro-photographic apparatus, bulb down, in such a way that the meniscus of the mercury in the space above the contraction could be photographed. The bulb was then heated until the mercury reached a given position in the capillary, and after the bulb was cooled, a photograph was taken of the position and shape of the meniscus. The tube was then inverted and another photograph taken. In the first case the force of the gravity is toward the contraction, while in the second case it is away from the contraction.

Two such photographs are shown in Fig. 4. "B" shows the mercury (black) in the tube and the general shape of the meniscus. The white space "A" represents the point "E" in Fig. 3. At "C" can be seen the outline of the constriction. Fig. 4 "a" is taken with the bulb down, and Fig. 4 "b" is taken with the bulb up. In order that the meniscus may be more easily compared, Fig. 4 "b" is inverted from the position

in which the exposure was originally made. The change in the position of the meniscus may be seen by comparing the two photographs. The dotted line in Fig. 4 "b" shows the position of the meniscus when photograph "a" was taken. This change in the position of the mercury represented about .9°F in this particular tube.

Other thermometers behave in an entirely different way. If the bulb be heated slowly the mercury rises in a perfectly uniform manner, and no "spits" are observable. Such thermometers often prove to be "retreaters," i.e., the mercury may run down from the capillary as the

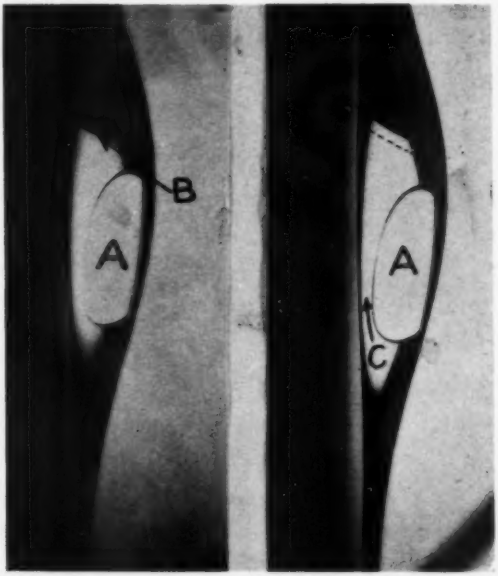

FIG. 4 (a) FIG. 4 (b)

bulb is cooled. In thermometers of this type, the break in the column usually occurs below rather than above the point of greatest constriction. Applying the theory previously outlined, these are thermometers where the shape above the contraction is such that the force of gravity pulling down more than overbalances any forces of surface tension tending to form drops. When such thermometers are cooled the mercury tends to retreat from the stem but the flow is regulated by its viscosity and the size of the constriction in the tube. If cooled rapidly enough, the mercury recedes into the bulb faster than it passes

through the contraction, and just below this constriction a state of tension actually exists. This causes a break in the column. If cooled slowly, the passage of mercury through the constriction is sufficiently rapid to prevent a break, and the thermometer exhibits retreat.

One of the causes of inaccurate thermometers is the presence of gas. A small amount above the mercury, in most cases, does no appreciable damage. If the bulb be heated and the mercury driven to the top of the tube, this gas, compressed into a volume many times smaller than its usual volume, may cause an otherwise satisfactory tube to retreat. Such a thermometer would be eliminated by a retreat test which will be described later. An amount of gas trapped in the mercury column may or may not be a source of error. A large amount, the length of of which is visible through the magnifying lens of the tube is liable to cause an error in reading equal to the length occupied by the air in the stem. Tubes showing such spaces should be rejected. Small spaces, i.e., spaces which do not cause a complete break in the mercury column but which cling to the side of the capillary are not liable to cause an appreciable error in reading and no reason exists for rejecting tubes showing such spaces.

The following paragraphs discuss specifications which is hoped might benefit the physician, the public and the manufacturer.

It is the problem of the manufacturer to produce goods which meet the standards of quality as determined by proper tests. The methods he develops and the apparatus designed for production are his own, and should not be subject to scrutiny or inspection by anyone without his consent.

The first requirement of a clinical thermometer is accuracy, and the following would appear to cover the case: "Thermometers shall be accurate to within .2°F throughout the scale, and the total error between any two test points four degrees apart shall not be more than .3°F. In successive tests at the same temperature the thermometer shall repeat its registration within .3°F. Any thermometer, showing gas in the bulb in large enough quantities to be visible to the eye shall be rejected. Gas in the capillary, below the top of the mercury, in sufficient quantities to prevent the uniting of the column is also grounds for rejection. A small amount of gas will be tolerated at the top of the column, provided the quantity is not sufficient to throw the index back more than 1/4 inch when tested for retreat as described in the second method given below."

The question of retreat is equally serious to the manufacturer. No test has yet been devised which will separate retreaters in the process of manufacture. A tube, which when completed shows retreat, must be practically remade, and can never fulfill the requirements for a first-class tube. This additional cost is added to the price of the finished tube and is paid for by the consumer.

At present two methods are used to test for retreat. One, which may be termed the "pull out," consists of gradually removing the thermometer from a bath and noting any movement of the index. This test closely approximates actual conditions and no thermometer should be considered as correct when the index recedes to such an extent as to cause the allowable error to be exceeded.

The second type of retreat test is now in use in one or more state laboratories. It has the advantage that large numbers of thermometers may be tested at one time. A number of tubes are put in a bath and heated until the mercury stands at the top of the column. The bath is then allowed to cool and tubes in which the index recedes more than 1/4 inch from the top of the bore, are considered retreaters. This, of course, does not even approximate actual conditions of use. It has been explained above that when a tube is allowed to cool a state of tension may actually exist in the mercury just below the construction. Whether or not a break occurs (as it should) depends upon the size and shape of the contraction, the viscosity of the mercury, and, finally, to a very considerable degree, on the rate of cooling. In this test, as now applied, very great differences may be expected in the rate of cooling as the test is applied from time to time. Consequently, tests have shown that the results obtained are not consistent. It is therefore necessary to specify the rate of cooling.

The physician removes the thermometer from the mouth of a patient at a temperature of at least 96°F. If the air in the room is 70°F the thermometer will fall about 26°F in two minutes. Hence, if specifications called for a test with the rate of fall of temperature one degree per minute, a factor of safety of around 1300% would be provided. This would appear sufficient. This method is more suitable for tests of large quantities at a time, and the results obtained are more consistent. It also appears somewhat more severe than the so-called "pull out test."

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A NOTE ON THE USE OF THE CONCAVE GRATING IN AN ASTIGMATIC MOUNTING

By Max Petersen

ABSTRACT

An arrangement is proposed whereby overlapping orders may be distingushed, using a simple quartz lense to project the image of the source on the slit of the spectrograph.

In the use of a grating in its higher orders the difficulty of distinguishing the lines of one order from those of another often prompts the use of an absorbing screen before the slit, eliminating one or more orders but sometimes weakening the light of the wanted order. When the elimination of line images is not particularly demanded and distinction between the different orders is all that is desired, a concave grating in a Rowland mounting permits another procedure which has been found so satisfactory as to seem worth description.

When a grating is mounted in divergent light, as in the Rowland manner, there are two astigmatic foci conjugate to the center point of the photographic plate. The inner focus, of which a point is conjugate to a line normal to the dispersion plane, is occupied by the slit; the outer, of which a point is conjugate to a line in the dispersion plane, is ordinarily not used. (Sirks[1] has pointed out its availability for an occulting screen or prism to permit juxtaposition of spectra or simultaneous exposure to two sources as is possible with stigmatic spectrographs. Dorgelo[2] remarks of the possibility of using a photometric wedge placed in this outer focus. When it is desired to record interferometer fringes the slit may itself be placed there, for with small angles of incidence the astigmatic width of the line images will not be intolerable.) For the promotion of accuracy and ease of measuring spectra against comparison lines of another source it has been found preferable completely to superpose the two spectra. Then to distinguish between their respective lines one may place in the outer focus a fine horizontal wire, which produces a sharp gap in the images of spectra projected past it. The vertical shift or removal of the wire will mark the lines of a particular exposure.

In the Rowland mounting the outer focus lies at the intersection of the collimation axis with the plane tangent to the center of the plate.

[1] J. L. Sirks, Astroph. Jour., *13*, p. 763.
[2] H. B. Dorgelo, Phys. Zs., *26*, 756; 1925.

When the grating is set at a large angle of incidence, as it will be for the higher orders, this outer focus may easily be farther from the slit than the lense which casts the source image on the slit. The overlapping orders may then be distinguished in the following manner: The lense used in front of the slit is of quartz, a simple double convex or a pair of plano-convex. The dispersion of the quartz, which is not objectionable for most purposes except, of course, the projection of interference fringes, is now put to use. Instead of putting the occulting wire at the outer focus it should be put in such a place that its virtual image formed by the projection lens will be there. Since the dispersion of the quartz makes this possible for but one wave length at a time the gaps in the lines of a chosen order will be sharp and those of other orders will be diffuse to a degree determined by the displacement of the virtual image for the wave lengths involved. As there is no preference which order has sharp line gaps there is almost no region in which the adjustment of the occulting wire cannot be visually accomplished.

A further feature of this virtual wire in the outer focus is that it would indicate Lyman ghosts, whose wave length would at once reveal itself.

The writer wishes to express his obligation to the University of Wisconsin, where the work was done which yielded this slight by-product.

WASHINGTON SQUARE COLLEGE,
NEW YORK UNIVERSITY,
FEBRUARY 23, 1926.

AN X-RAY SPECTROGRAPH

By John T. Norton*

ABSTRACT

An x-ray spectrograph of simple construction but of considerable precision is described in detail. It is of the rotating crystal type and the record is made upon a photographic film bent in the arc of a circle. The instrument is very easy to set up and the precision of the results is quite good for a device of this type.

The rotating crystal type of x-ray spectrograph is very useful for the measurement of the spacings of crystals in directions normal to cleavage faces as well as for determining the wave lengths of emission lines, absorption edges and short wave limits. An instrument has been designed for this work which is of quite simple construction, and while it involves no new principles, it has a number of features which make it very convenient to set up and use. It seems best to describe the various parts separately.

Base. The base of the instrument is made of a heavy brass casting supported on three levelling screws. A hole in the center is bored out to receive the rotating crystal holder, and the top of the base machined at the same time so that it is accurately at right angles to the axis of the hole. On the front end of the base casting is screwed the lead shield which supports the slit system and at the back are three holes into which pins on the bottom of the cassette fit.

The Rotating System. Into the hole in the center of the base is fitted a sleeve with a shoulder at the top and a nut and lock nut at the bottom. This is made a good running fit and to the bottom nut is rigidly attached the cam follower. The center of the sleeve is bored out to a standard taper and into this fits the crystal holder. It is essential that the inside and outside of this rotating sleeve be concentric. The sleeve is rotated by a heart shaped cam designed to give a uniform angular velocity and the method of designing such a cam has been described by Friauf.[1] For most work, however, the cam need not be very precisely formed. Three sizes of cam have been found sufficient, giving the crystal an angle of about 30, 10 and 5 degrees rotation respectively. The cam is driven by a double worm gear reduction, the ratio of which is 4000

* Assistant Professor of Physics, Mass. Inst. of Technology.

[1] J. B. Friauf—Design of a Cam for an x-ray Spectrograph. J. O. S. A. & R. S. I. *11*, p. 289; 1925.

to 1. The first worm is driven by a coil spring belt from a small variable speed motor. The crystal makes a complete cycle in about 5 minutes.

The Crystal Holder. The crystal holder is designed so that a crystal can be mounted and set at the proper angle quickly and accurately. It consists of a taper shank which fits into the rotating sleeve, a graduated disc, 3 inches in diameter, divided in single degrees, and a frame of bakelite which is to support the crystal. This frame which will take a crystal as large as .75 in. square is made so that one face is accurately in the axis of the taper shank. It is machined on the centers used for making the finishing cut on the taper, and with care it may be done quite accurately. Bakelite is used for this frame to prevent any diffraction effects. To mount the crystal, it is merely necessary to clamp a small piece of plane glass to the surface of the frame, place the cleavage face of the crystal in contact with the glass and pour in melted paraffin. The crystal holder may then be put in place and adjusted to the proper angle.

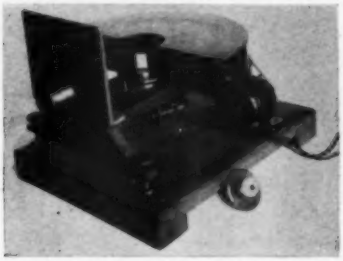


FIG. 1. *Spectrograph with shield removed showing crystal holder and cassette.*

The Slit System. The slit system is supported by the lead shield which forms the front of the instrument. This shield is made of one-quarter inch sheet lead, and in the center is soldered a brass tube. Into this telescopes another tube containing the three slits. The slit nearest the x-ray source is fastened to the end of this tube. The second slit is fastened to a short piece of tubing which may be rotated inside of the others so that the slits may be made parallel. These slits are of a fixed width of .015 inches, and are of brass. A third, adjustable slit is used to cut off the diffracted beams from the second slit. This slit system gives excellent sharp lines with very little scattering.

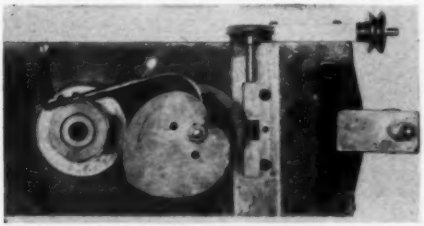


FIG. 2. *Under side of base showing arrangement of cam and worm gears.*

The Cassette. The cassette was made so that it might be separated from the rest of the apparatus for filling and yet have a perfectly rigid position when in place. This was accomplished by placing three steel pins one-quarter-inch in diameter in the bottom of the cassette which fit accurately in holes in the base, and the cassette is supported only at these three points. The cassette is semi-circular in form, and is made of three-ply wood. The film rests against a black celluloid strip and is held in place by a felt covered spring of phosphor bronze which makes a light-tight joint. The radius of the cassette is six inches. When in use the cassette is covered with a lead shield to prevent fogging of the film by stray radiation.

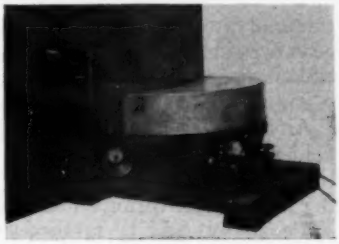


FIG. 3. *Spectrograph with shields in place ready to run.*

This instrument has been in use for about a year, largely for instruction work, and has given very good results. Using the molybdenum target water-cooled tube running at 20 milliamperes, it is possible to obtain a film showing the K series lines in four orders with a rock salt crystal in about two hours. The removable crystal holder is very convenient, and it is possible to mount two crystals in this holder, one above the other, and obtain the two patterns on the same film. The accompanying figures show some of the details of the instrument and the way in which it is set up.

DEPT. OF PHYSICS,
MASS. INST. OF TECH.,
CAMBRIDGE, MASS.

Practical Photo-Micrography. By J. E. Barnard, F.R.S., Lecturer in Microscopy, King's College, London, and Frank V. Welch. xii+316 pages and 16 plates. Longmans, Green and Company, New York. $6.00.

The present volume is the second edition of a work which appeared originally in 1911. As the authors state, there have been no outstanding advances in the subject since the first edition. Nevertheless, there have been certain changes, both of apparatus and method, and the present edition aims to bring the subject up to date, particularly as to method.

Any treatise which attempts to cover in detail the whole of this vast field must of necessity be very extensive. For in addition to covering the principles, apparatus, and methods of both microscopy and photography, descriptions must be given of procedure peculiar to the subject itself and to methods of preparation of specimens in widely different fields, such as for example metallography and bacteriology. The present volume makes no attempt at an exhaustive treatise, but gives a non-technical outline, founded on the authors' own practical experience and intended to "enable another person to carry out" the various methods described.

The content of the book is best given by the chapter headings: The Microscope; The Optical Equipment; Objectives and Oculars; The Optical Equipment: Condensers and Collecting Lenses; Sources of Illumination; The Photo-Micrographic Camera; The Use and Manipulation of the Microscope; General Preliminary Preparations; Color-screens, Plates, Exposure; Photographic Processes; Special Processes; Examples of Photo-micrographic Technique.

In view of the necessity for brevity, the book would have been improved by more frequent references to original sources. Furthermore, the reviewer is disposed to suggest that in view of the almost universal availability of literature descriptive of the general principles of microscopes and of photographic processes, it might have been wiser to have used the valuable space in the book for describing processes peculiar to photo-micrography, such as preparation of metal surfaces for metallography, than to have taken an entire chapter, for example, to describe the development of a photographic negative and the making of lantern slides. Nevertheless, the book *is* a reasonably complete survey of the field, and the critical student will have no difficulty in finding elsewhere whatever supplementary literature he may need. And the most excellent and well chosen examples of photomicrographic art, some 43 in number covering a wide variety of subjects, make eloquent testimony to the fact that the authors write from first-hand knowledge, which, after all, is a most excellent recommendation for a book of this kind.

F. K. RICHTMYER

A HIGH-VACUUM X-RAY SPECTROMETER

By Manne Siegbahn and Robert Thoræus

INTRODUCTION

The spectrometer whose design and operation will be given in this paper is intended for general x-ray spectroscopic purposes where fairly high precision is wanted. It is, however, not constructed with the view of making absolute measurements of x-ray wave lengths. A sufficient number of x-ray lines distributed over the whole wave-length domain

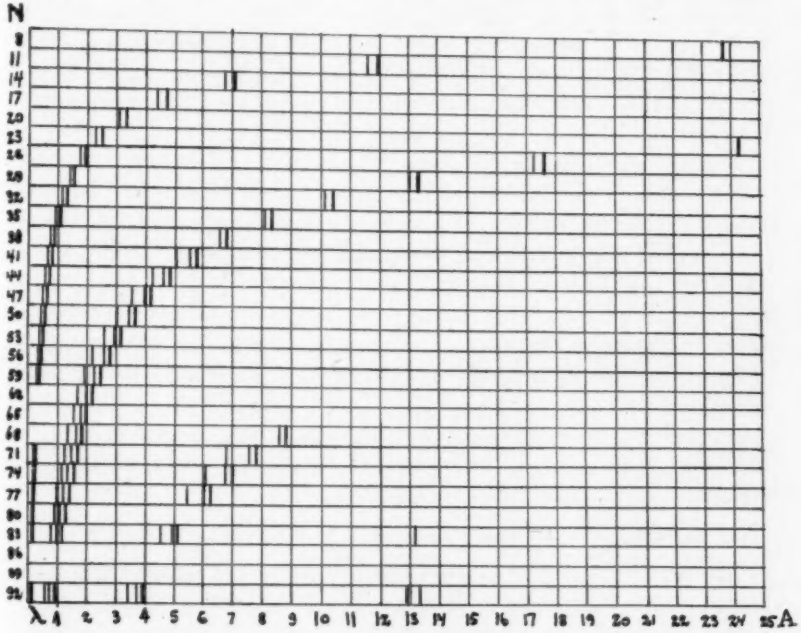


Fig. 1

is now available and so an instrument for measurements relative to these known lines will serve most spectroscopic purposes and be all that is needed.

Concerning the domain of wave-lengths which must be attainable by an x-ray spectrometer, a glance on the diagram in Fig. 1 may give information. By an ordinary x-ray tube only the small spectral region

to the left of the vertical line corresponding to about 1 ÅU. is emitted, the remainder being absorbed by the walls of the tube. By supplying the tube with a window of a thin aluminium-foil, 2 or 3 ÅU. may be attained by a spectrometer in air of atmospheric pressure.

As may be seen from the diagram, however, the major portion of the known x-ray spectra would be out of observation under such conditions. To include also these regions it is necessary to build the spectrometer in such a manner that the absorption along the path of the x-ray is sufficiently reduced. In former instruments this requirement has been provided by attaching a specially built x-ray tube to a spectrometer in which the air-pressure could be reduced to some millimeters of Hg, the spectrometer chamber then being separated from the x-ray tube only by a very thin foil of aluminium, gold-beaters skin, celluloid. etc. In this way Moseley was able to detect x-ray spectral-lines up to about 8 ÅU. In the vacuum spectrometers constructed by one of us about 12 ÅU. was reached.

A step further in the direction of long wave lengths was taken by using a slightly modified spectrometer that could be evacuated to the same very low pressure as the tube itself.[1] In this case the foil between the tube and the spectrometer was inserted only to prevent an undesired blackening of the photographic plate by the ordinary light coming from the tube. As this foil is not required to stand any pressure or to prevent leakage of air from the spectrometer to the tube it could be used much thinner than in the former instruments. By using an oxide-cathode instead of the tungsten hot-wire the emission of light having photographic effect may be sufficiently decreased to permit the complete omission of the foil. In this way wave-lengths of more than 20 ÅU. have recently been registered and measured.[2]

DESCRIPTION OF THE NEW HIGH-VACUUM SPECTROMETER

The spectrometer herein described is constructed with the view of allowing also these long wave-lengths to be studied in a convenient way so that, with one and the same instrument, practically all the x-ray spectra may be photographed and measured with considerable accuracy. An important point in constructing this spectrometer is the fact that, by choosing suitable crystal-gratings, it is quite possible to cover the desired range of wave-lengths within reflection-angles of 35° maximum. Such crystal-gratings are for instance

[1] Siegbahn—Thoræus: Arkiv f. mat. astr. o Fys. Bd., 19A. Nol 12; 1925.
[2] Thoræus: Phil. Mag.; Feb. 1926.

Calcite	2d = 6.058 ÅU.
Mica	2d = 19.854 ÅU.
Palmitic acid	2d = 70.98 ÅU.

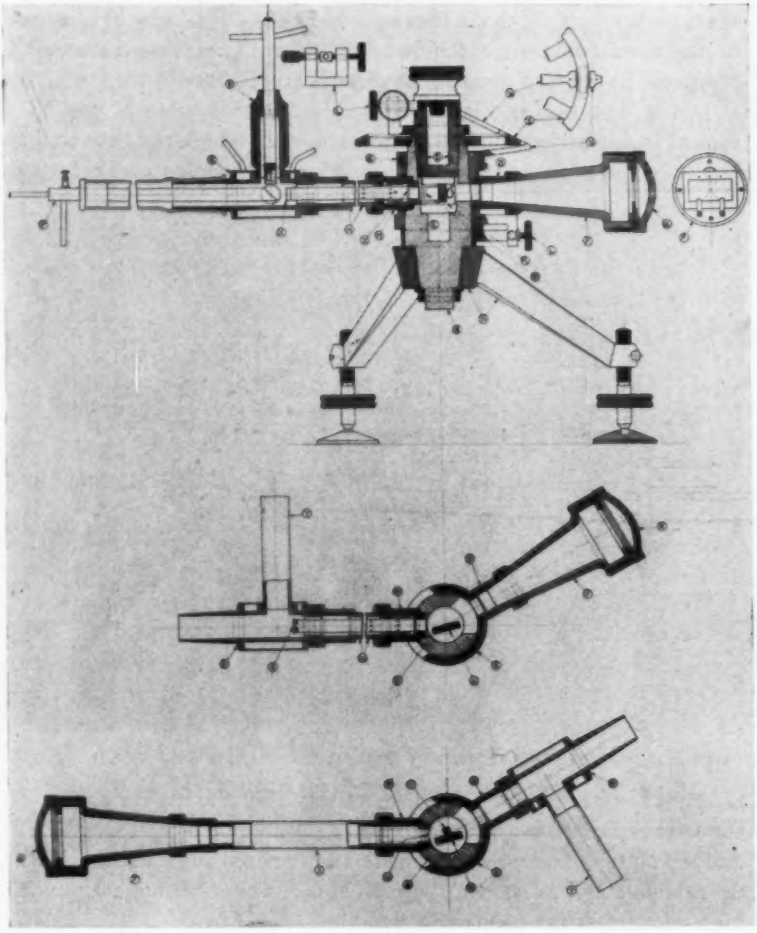


FIG. 2

The new spectrometer which is shown in figs 2 and 3 has some features in common with the Schumann ultraviolet one. The central part of it consists of a steel cylinder B closed at the bottom and mounted on a

tripod A in such a way that it can be rotated on a vertical axis. The upper two thirds of the steel rod is conically turned on its outer surface. A brass cylinder C, carrying a side tube N, is fitted on this steel conic. On the wall of the brass-cylinder, opposite the side-tube N, there is an aperture through which another tube M passes. This tube M is screwed to the steel conic after the brass-cylinder C has been mounted in position. In the wall opposite M there is an aperture in the steel conic, as will be seen best in Fig. 2b or c. The two side-tubes M and N are conical inside and ground to a standard size. The x-ray tube and the holder of the photographic plate are by corresponding conics fastened

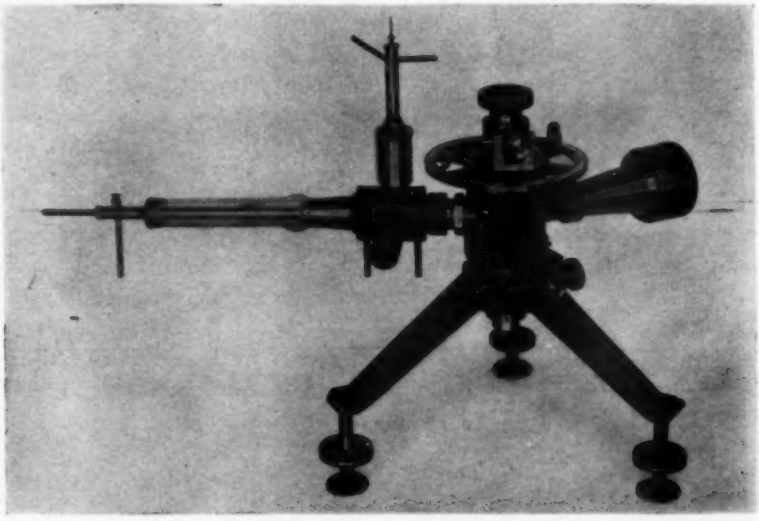


FIG. 3

air-tight to the two side-tubes respectively. By making the aperture in the steel conic and the brass cylinder of suitable shape and size the tube and the plate-holder may be turned to any desired angle with each other up to about 35°, as shown in fig. 2b and c, while still maintaining the high vacuum in the spectrometer.

The reflecting crystal K is mounted on a central brass conic D which fits in the upper inner part of the steel conic B. As this is an air-tight connection, it permits the rotation of the crystal through the vertical axis of the spectrometer. To enable a rapid changing of crystals and to secure an immediate adjustment of the reflecting surface parallel to, and

in, the vertical axis there is a brass clamp Pf attached to the bottom of the inner brass-conic in such a way that the upper and lower fronts of the clamp are just in the axis. The crystals (of standard dimensions) are pressed against the clamp fronts by two springs. A small cylinder Pl, with apertures only where the rays have to pass through it, is put over the crystal to protect it.

There is further a circular scale S fixed to the upper part of the steel conic. The two movable parts, the outer and the inner brass-conics, have one pointer each, V_1 and V_2, showing on the scale and thus allowing the relative positions of the different parts of the spectrometer to be read. There are also on each of the movable parts micrometric devices L which permit small and exact rotations.

The holder F of the photographic plate does not need detailed explanation. There is an air-tight conic cap G on its back-side for inserting the plate and a fix cross-wire is arranged to give a zero-mark on the spectral-plate.

The metallic x-ray tube R is of our ordinary design with interchangeable hot-cathode O and anticathode P, the latter being insulated by a standard glass conic. Tube, cathode and anticathode are cooled by water (or oil for the anticathode). On the x-ray tube there is a wide side-tube T which serves for evacuating the system: spectrometer—x-ray tube. As we always use rotating molecular pumps we had a flexible metallic tube[3] soldered to the tube T. This flexible tube, on its other end, had a metal conic fitting to the pump.

When the spectrometer is to be used with an ionisation chamber instead of the photographic registering arrangement the plate holder F is replaced by a tube provided with a slit in its outer end. This slit is covered by a thin foil and then the ionisation-chamber is screwed to this end of the last-mentioned tube. The same arrangement is applicable to the Geiger-method for counting x-ray quanta. A description of such an arrangement and the results obtained thereby will appear within a short time. Also a device to use the photoelectric effect of the radiation for its registration may easily be applied in a similar way.

In spite of the fact that the system, spectrometer—x-ray tube, had 8 or 9 conic joints and 2 cemented by pizein (the outer ends of cathode and anticathode) there were no difficulties in getting the necessary high vacuum. The time from starting the pumps to the moment when the tube with cold cathode stood the high-tension without discharge was about 5 min.

[3] Supplied by the Berlin—Karlsruher Industrie Werke A-S, Karlsruhe, Germany.

How to use the instrument

There are principally three different spectroscopic methods which may all be applied on this spectrometer simply by changing the connections of the tubes. The three methods are as follows:

1. Slit in front of the crystal.
2. Slit on the crystal.
3. Slit behind the crystal.

In method 1) it is advantageous to have the slit and the spectral-plate both at the same distance from the rotating axis of the crystal, as was first pointed out by Bragg. A second condition which is important in each of the three methods is to make the distance from the x-ray source to the slit (or crystal) as short as possible.

How the instrument is arranged for method 1), which is the most universal one and which will be the ordinary way of using is shown in Fig. 2a and b. The x-ray tube R is, by an inserted tube H, connected to the side tube M of the steel conic. The slit X is mounted on a cylinder placed inside the tube H in such a way that the distance from the slit to the anticathode is only a few millimeters. There are further some holes in this cylinder to permit the air free passage which the slit itself is not able to do because of its being only 1/10 of a millimeter wide.

In the method 1) the plate-holder F is connected to the side-tube N of the outer brass-cylinder C. The distance from the axis of the spectrometer to the plate should then be equal to the distance from the slit to the axis. In all three methods, however, the dispersion can be varied by inserting extra tubes between the plate-holder or x-ray tube and the central part of the spectrometer. Without any extra tubes before the plate-holder the distance from the slit to the photographic plate in our instrument is 360 mm. This gives a suitable dispersion for most purposes (slit-width 0,1—0,05 mm.)

To prevent the ordinary light from the x-ray tube from reaching the plate-holder it is quite sufficient to cover the apertures in the cylinder R with a sheet of thinnest aluminium foil or colored celluloid film.

The rearrangement of the instrument when used according to the methods 2) and 3) is shown in Fig. 2c. In this case the x-ray tube is connected to the brass tube instead of to the steel conic. To get the x-ray tube as near the crystal as possible it is arranged directly on the side tube. The plate-holder is then attached to the side tube M of the steel conic.

In method 2) a steel wedge is mounted at the front of the crystal surface in its axis of rotation, thus forming together with its image in the reflecting surface a slit the width of which can be varied by putting foils of suitable thickness between the edges of the wedge and the crystal. In method 3) the slit is placed in the position U.

As the dispersion is determined by the distance from the slit to the spectral plate the total length of the plate-holder tube in methods 2) and 3) must be about twice that in method 1.)

On account of details of construction it is possible in method 1) to get the x-ray source nearer the slit than in the two other methods, and because of this the first method is the most efficient one, i.e. will give shorter exposure-times.

In all methods only a limited region of wave-lengths may be registered by one exposure. These limits are given by the length of the spectral plate which in our instrument was about 6 cm. (plate-size 4.5 x 6 cm.). When working according to the method 1) x-ray tube and plate-holder are in fixed positions and the crystal-holder is turned over a suitable range of angles. In methods 2) and 3) the plate-holder and the crystal-holder are fixed and the x-ray tube should be turned through the required range of angles. Now, however, as the x-ray tube is stationary because of its connections to the pump, this relative motion is carried out in such a way that the spectrometer is rotated in the tripod.

SOME SPECTROGRAMS OF LONG WAVE-LENGTHS

Finally we supply some spectrograms to illustrate the working of the instrument. The three spectrograms Fig. 4a show the K series of aluminium ($\lambda = 7.94 - 8.32$ ÅU.) photographed according to methods 1),2) and 3) and all of them with the same dispersion. As grating, a mica-crystal was used. The width of the slit was 0.1 mm and the distancefrom slit to plate was 360 mm. The first spectrogram of Fig. 4b is the same K series taken with greater dispersion obtained by making the distance slit to plate 1118 mm. The two spark lines are here fairly separated.

Fig. 4b also shows the spectrograms at very long wave lengths, giving respectively 15.94 and 21.5 ÅU. The first one is the L series (α and β lines) of Cobalt and the second one that of Chromium. These two spectrograms were taken with palmitic acid as grating. In this case the organic substance was applied in a very thin layer on a glass-

plate either by melting or by crystallization of a solution according to methods given by Müller and Trillat.

As mentioned above the different parts of the known x-ray spectral region are attainable by using some one of the three crystals: Calcite,

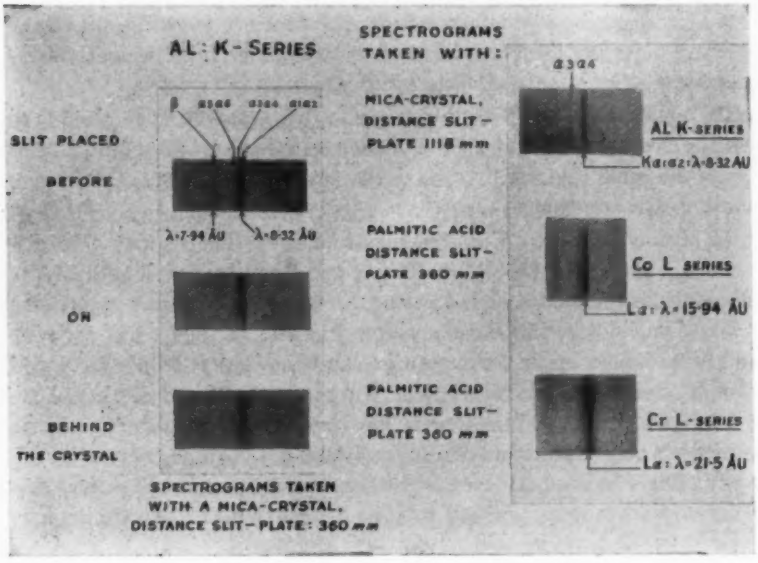


FIG. 4

Mica and Palmitic acid. On our instrument the reflecting angles can be varied from 0° to 35° corresponding to the following wave-lengths:

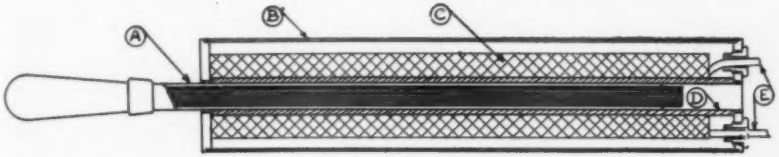
Calcite	<3.5 ÅU.
Mica	<11.0 ÅU.
Palmitic acid	<40.0 ÅU.

PHYSICAL LABORATORY OF THE UNIVERSITY, UPSALA, SWEDEN.
APRIL, 1926.

A CHOKE COIL FOR CONTROLLING FILAMENT CURRENT IN X-RAY TUBES*

By Arthur E. Kunberger and Charles E. Lanyon

Much trouble has been experienced in this Laboratory with the usual type of coiled nichrome wire resistance used for regulating current through filaments of x-ray tubes. After continuous operation, as for example in commercial work, the resistance wire oxidizes, causing poor contact between the movable finger and the wire, resulting in arcing and consequent pitting. The finger then heats up, losing its temper and thereby making the contact still worse. After this has taken place we have noticed fluctuations of as much as seven milliamperes in the current flowing in the tube circuit. In order to maintain constant



Choke Coil
A—*Adjustable iron core.*
B—*Brass case.*
C—*Copper wire winding.*
D—*Fiber tube.*
E—*Connecting wires.*

exposure conditions it is just as essential to regulate the current as the time. Hence the need of a dependable, easily adjusted and permanent piece of equipment at this point can be well appreciated.

Inasmuch as the filament current is usually supplied from a transformer the idea of using a choke coil suggests itself. One was designed and built consisting of about one and one-half pounds of No. 15 double covered wire on a fiber tube five-eights inch in diameter and seven inches long. For neatness and convenience in mounting it is enclosed in a brass case, through insulators in which the connecting wires are brought. To avoid the transformer action set up if the case forms a closed circuit around the windings care must be taken to split the brass

* Contribution from the x-ray Division, Research Laboratory of Applied Chemistry, Mass. Inst. of Technology, Cambridge, Mass.

shield. The iron core is made of a bundle of soft iron wires packed into a split brass tube which slides tightly inside the fiber tube. The ends of the two wires from the coil are connected to the same two points as was the rheostat. Adjustment of the current is made by pushing or pulling the iron core in or out of the coil. With the coil constructed as above for use with twelve volts, regulation from one-half ampere in the primary circuit with the core all the way in, to eight amperes with the core all the way out, can be obtained.

The range of adjustment is large, the regulation is fine and can be made continuously instead of in steps as in a wire rheostat. No sliding contacts are employed so that falling off in efficiency cannot take place. The regulator is not mounted on the end of the x-ray tube and strains in the tube while adjusting the current are avoided.

The best place for mounting the coil has been found to be next to the stabilizer on top of the transformer so that all the high tension leads are totally enclosed. Even in this position, where there is little ventilation, the rise in temperature is only about 40°*C* above room temperature. After some 2500 hours of practically continuous operation it has given entire satisfaction.

Correlations Between Stark Effect and Other Qualities of Lines.—Stark compiles data upon the effect of electric field on series-lines of the types (2*p-ms*), (2*p-md*), (2*p-mf*) and (2*p-mg*), taking as particular measure of this effect the wave length difference between the position of the line when uninfluenced by field, and its position (or that of its component of greatest wave length) when shifted (or shifted and resolved) by electric fields of 28,000 volts/cm. This quantity turns out to be fairly closely, though not perfectly, correlated with the difference between the second term in the foregoing expression for the line, and the hydrogen terms R/m^2. When the term in question is greater (less) than the hydrogen term, the component of greatest wave length in the electric field is greater (less) in wave length than the unshifted line; and usually, the greater the difference between the term in question and the hydrogen term, the smaller is the shift. In making these correlations Stark uses chiefly the values of m originally attached to the various levels, not those more recently introduced by Bohr and others.—[J. Stark (Ullersricht bei Weiden); Ann. d. Phys., *78*, pp. 425–433; 1925.]

KARL K. DARROW